

# Calabi-Yau structures and Einstein-Sasakian structures on crepant resolutions of isolated singularities

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## Abstract

Let  $X_0$  be an affine variety with only normal isolated singularity  $p$  and  $\pi : X \rightarrow X_0$  a smooth resolution of the singularity with trivial canonical line bundle  $K_X$ . If the complement of the affine variety  $X_0 \setminus \{p\}$  is the cone  $C(S) = \mathbb{R}_{>0} \times S$  of an Einstein-Sasakian manifold  $S$ , we shall prove that the crepant resolution  $X$  of  $X_0$  admits a complete Ricci-flat Kähler metric in every Kähler class in  $H^2(X, \mathbb{R})$ . We also obtain a uniqueness theorem of Ricci-flat conical Kähler metrics in each Kähler class with a certain boundary condition. We show there are many examples of Ricci-flat complete Kähler manifolds arising as crepant resolutions.

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## Introduction

Let  $X$  be a Kähler manifold of complex dimension  $n$  with trivial canonical line bundle  $K_X$  and  $\Omega$  a nowhere vanishing holomorphic  $n$ -form on  $X$ . If a Kähler form  $\omega$  satisfies the following equation,

$$\Omega \wedge \overline{\Omega} = c_n \omega^n,$$

for a constant  $c_n$ , then the Ricci curvature of  $\omega$  vanishes, that is,  $\omega$  is a Ricci-flat Kähler metric, where  $\overline{\Omega}$  is the complex conjugate of  $\Omega$ . The well-known Calabi-Yau theorem, due to Yau on a compact Kähler manifold with the first Chern class  $c_1 = 0$  was proved by solving the Monge-Ampère equation which shows that there exists a unique Ricci-flat Kähler metric in each Kähler

class. On a non-compact complete Kähler manifold  $X$  with  $c_1(X) = 0$ , it is an outstanding problem whether there exist Ricci-flat Kähler metrics. We need to impose suitable asymptotic conditions on the boundary. There are many remarkable results on Calabi-Yau theorem on non-compact complete Kähler manifolds with  $c_1 = 0$ . Tian and Yau [26], [27], Bando-Kobayashi [3] and Joyce [15] solved the Monge-Ampère equation under various boundary conditions. On the other hand, the hyperKähler quotient construction [14], [16] produces many Ricci-flat Kähler manifolds in a simple and algebraic way, some of which are not obtained by the analytic method [12]. Recently rapid developments occur in the Einstein-Sasakian geometry which yield a new view point of the problem of Ricci-flat Kähler metrics. If we have a positive Einstein-Sasakian manifold  $S$ , then the cone  $C(S) = \mathbb{R}_{>0} \times S$  admits a Ricci-flat Kähler cone metric. Boyer-Galicki [4] constructed a family of positive Einstein-Sasakian metrics on the links on hypersurfaces with isolated singularities, which includes interesting examples such as homology spheres. Futaki, Ono and Wang [10] showed that a sphere bundle of the canonical line bundle on every toric compact Fano manifold admits an Einstein-Sasakian metric. These results imply that there are many notable examples of Ricci-flat cone metrics which are constructed by Einstein-Sasakian manifolds. In the present paper, we introduce conical Kähler metrics which are complete Kähler metrics with the certain boundary condition (see the definition 1.3 in section 1). We apply an existence theorem of Ricci-flat Kähler metrics to the class of conical Kähler metrics. Let  $\omega$  be a conical Kähler metric on  $X$  with  $\Omega \wedge \overline{\Omega} = c_n F \omega^n$  for a positive function  $F$ . If  $F$  satisfies

$$\|e^{(2+\delta)t}(F-1)\|_{C^k} < \infty,$$

for  $0 < \delta < 2n - 2$  and  $k \geq 2$ , then there exists a Ricci-flat conical Kähler metric  $\omega_u$  on  $X$  (see the theorem 1.5 in section 1 for more detail).

The existence theorem can be also deduced from the arguments by Bando-Kobayashi in [3]. Our boundary conditions in the theorem are modified for conical Kähler manifolds. The author gives a proof of the theorem for the sake of readers.

As an application, we discuss the existence of Ricci-flat Kähler metrics on resolution  $X$  with trivial  $K_X$  of an affine variety  $X_0$  with only normal isolated singularity  $\{p\}$ .

**Theorem 5.1** *Let  $X_0$  be an affine variety with only normal isolated singularity  $\{p\}$ . We assume that the complement  $X_0 \setminus \{p\}$  is biholomorphic to the cone  $C(S)$  of an Einstein-Sasakian manifold  $S$  of dimension  $2n - 1$ . If there is a resolution of singularity  $\pi : X \rightarrow X_0$  with trivial canonical line bundle  $K_X$ , then there is a Ricci-flat complete Kähler metric for every Kähler class of  $X$ .*

Our theorem covers the crucial case where a Kähler class does not belong to the compactly supported cohomology group. We use a vanishing theorem on  $X$  and the Hodge and the Lefschetz decomposition theorems on a Sasakian manifold to construct a suitable initial Kähler metric in every Kähler class which the existence theorem can be applied. We show that a Ricci-flat Kähler conical metric is unique in each Kähler class if we impose a certain boundary condition on metrics. (see the theorem 1.8).

In section 1, we introduce the class of conical Kähler metrics and show the existence theorem and the uniqueness theorem of Ricci-flat Kähler metrics on them. In section 2 we give a proof of the existence theorem and the uniqueness theorem. In section 3, we will give a short explanation of Sasakian metrics and Kähler cone metrics. In section 4 we discuss the one to one correspondence between Einstein-Sasakian structures and Ricci-flat Kähler cone metrics. In section 5, as an application of section 1, we obtain Ricci-flat Kähler conical metrics on crepant resolutions of normal isolated singularities as above. In section 6, we construct several families of Ricci-flat Kähler conical metrics on crepant resolutions of normal isolated singularities. Our examples include: resolutions of the isolated quotient singularities, the total spaces of canonical line bundles of Kähler-Einstein Fano manifolds, the total space of the canonical line bundle of every toric Fano manifold and small resolutions of ordinary double points of dimension 3

Some of these examples are already known. Joyce [15] showed the Calabi-Yau theorem on resolutions of the isolated quotient singularities which is called asymptotically locally Euclidean (ALE). Calabi [5] used the bundle construction to obtain Ricci-flat Kähler metrics on the total spaces of the canonical line bundles of Kähler-Einstein Fano manifolds and Futaki [9] also constructed Ricci-flat Kähler metrics on the total spaces of the canonical line bundles of toric Fano manifolds. The Kähler classes of these Ricci-flat Kähler metrics lie in the compactly support cohomology group and van Coevering [7] also constructed Ricci-flat Kähler metrics on crepant resolutions whose Kähler classes belong to the compactly support cohomology group.

Our method provides wider classes of complete Ricci-flat Kähler metrics (see the example in section 6) and the theorem 5.1 shows that the conjecture discussed in [20] and [7] on the existence of complete Ricci-flat Kähler metrics on resolutions of cones is affirmative.

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# 1 Existence theorem of Ricci-flat conical Kähler metrics

Let  $(S, g_S)$  be a compact Riemannian manifold of dimension  $2n - 1$  and  $C(S) = \mathbb{R}_{>0} \times S$  the product of the positive real number  $\mathbb{R}_{>0}$  and  $S$  with a coordinate  $r \in \mathbb{R}_{>0}$ , which is called the cone of  $S$ . By changing the coordinate  $t = \log r$ , we regard the cone  $C(S)$  as the cylinder  $\mathbb{R} \times S$  with the cylinder parameter  $t \in \mathbb{R}$

**Definition 1.1** *The cylinder metric  $g_{\text{cyl}}$  on  $C(S)$  is the product metric,*

$$dt^2 + g_S$$

*and the cone metric  $g_{\text{cone}}$  is given by*

$$g_{\text{cone}} = dr^2 + r^2 g_S.$$

*Let  $\nabla_{\text{cyl}}$  be the Levi-civita connection with respect to the cylinder metric  $g_{\text{cyl}}$  and  $|\alpha|_{g_{\text{cyl}}}$  the point-wise norm of a tensor  $\alpha$  by  $g_{\text{cyl}}$ . The  $C^k$ -norm of a tensor  $\alpha$  is given by*

$$\|\alpha\|_{C^k} = \sum_{i=0}^k \sup |\nabla_{\text{cyl}}^i \alpha|_{g_{\text{cyl}}}.$$

*We also have the Hölder norm  $\|\alpha\|_{C^{k,\alpha}}$ , for  $0 < \alpha < 1$ . In this paper we use the  $C^k$ -norm and  $C^{k,\alpha}$ -norm with respect to the cylinder metric  $g_{\text{cyl}}$  unless it is mentioned.*

Since  $r = e^t$ , we have the relation

$$g_{\text{cone}} = r^2 g_{\text{cyl}} \tag{1.1}$$

which follows from

$$\begin{aligned} r^2 g_{\text{cyl}} &= r^2 (dt)^2 + r^2 g_S \\ &= (dr)^2 + r^2 g_S \\ &= g_{\text{cone}}. \end{aligned}$$

**Definition 1.2** *A manifold  $X$  has a cylindrical boundary if there is a compact set  $K$  of  $X$  such that the complement  $X \setminus K$  is diffeomorphic to the*

cylinder  $C(S) = \mathbb{R} \times S$ . We identify  $X \setminus K$  with the cone  $C(S)$ . A Riemannian metric  $\bar{g}$  on a manifold  $X$  with a cylindrical boundary  $C(S)$  is a cylindrical metric if  $\bar{g}$  satisfies the following condition on  $X \setminus K \cong C(S)$ ,

$$\|e^{\delta t}(\bar{g} - g_{\text{cyl}})\|_{C^k} < \infty,$$

for some  $\delta > 0$  and an integer  $k > 4$ . In other words, the difference between  $\bar{g}$  and  $g_{\text{cyl}}$  decays exponentially with order  $O(e^{-\delta t})$ , including their higher order derivatives up to  $k$

$$\sum_{i=0}^k |\nabla_{\text{cyl}}^i(\bar{g} - g_{\text{cyl}})|_{g_{\text{cyl}}} = O(e^{-\delta t}),$$

We always extend the cylinder parameter  $t$  as a  $C^\infty$  function on  $X$  and identify the complement  $X \setminus K$  with the cone  $C(S)$ .

**Definition 1.3** A Riemannian metric  $g$  on a manifold with a cylindrical boundary  $C(S)$  is a conical metric if  $r^{-2}g = e^{-2t}g$  is a cylindrical metric, that is,  $g$  satisfies

$$\|e^{(-2+\delta)t}(g - g_{\text{cyl}})\|_{C^k} < \infty,$$

A conical Kähler metric on a complex manifold  $X$  is a conical Riemannian metric which is Kählerian. A conical Kähler form  $\omega$  is a Kähler form with the associate Riemannian metric is conical and a manifold with a conical Kähler form is called a conical Kähler manifold.

**Definition 1.4** Let  $(X, \omega)$  be a conical Kähler manifold with trivial canonical line bundle  $K_X$  and  $\Omega$  a nowhere vanishing holomorphic  $n$ -form  $\Omega$  on  $X$ . If a pair  $(\Omega, \omega)$  satisfies the following equation

$$\Omega \wedge \bar{\Omega} = c_n \omega^n,$$

for a constant  $c_n$ , then  $(\Omega, \omega)$  is called a Calabi-Yau structure whose Kähler form  $\omega$  gives the Ricci-flat Kähler metric.

The following theorem can also deduced from the arguments by Bando-Kobayashi in [3].

**Theorem 1.5** Let  $(X, \omega)$  be a conical Kähler manifold of complex dimension  $n$  with trivial canonical line bundle  $K_X$  and  $\Omega$  a nowhere vanishing holomorphic  $n$ -form  $\Omega$  on  $X$  which defines a positive function  $F$  by

$$\Omega \wedge \bar{\Omega} = c_n F \omega^n.$$

If  $F$  satisfies the following condition

$$\|e^{(2+\delta)t}(F-1)\|_{C^{k,\alpha}} < \infty,$$

for  $0 < \delta < 2n-2$  and  $k \geq 2$ ,  $0 < \alpha < 1$ , then there exists a smooth solution  $u$  of the Monge-Ampère equation  $\Omega \wedge \bar{\Omega} = c_n \omega_u^n$ , such that

$$\omega_u = \omega + \sqrt{-1} \partial \bar{\partial} u \quad (1.2)$$

is a conical Ricci-flat Kähler form with the condition,

$$\|e^{\delta t} u\|_{C^{k+2,\alpha}} < \infty.$$

(Note that  $\omega_u$  is a complete Ricci-flat Kähler metric.)

**Remark 1.6** In the theorem 1.5, the decay order of  $F-1$  is crucial for the existence of Ricci-flat Kähler metrics. Note that we estimate it by the cylinder metric  $g_{\text{cyl}}$ . In order to obtain a solution of the Monge-Ampère equation, we need to solve the equation of the Laplacian with respect to the conical metric  $g$ ,  $\Delta_g u = v$  on  $X$ , which is a linearization of the Monge-Ampère equation. The equation  $\Delta_g u = v$  has a unique solution  $u$  with the decay order  $O(e^{-\delta t})$  if  $v$  decays with order  $O(e^{-(2+\delta)t})$  for  $0 < \delta < 2n-2$ . Thus we require that  $F$  decays with the order  $O(e^{-(2+\delta)t})$ .

We show that a Ricci-flat conical Kähler metric in the form as in theorem 1.5 is unique.

**Theorem 1.7** If there are two Ricci-flat conical Kähler metrics  $\omega$  and  $\omega'$  satisfying

$$\omega' = \omega + \sqrt{-1} \partial \bar{\partial} u,$$

where  $u$  is a function with  $\|e^{\delta t} u\|_{C^k} < \infty$ , for a positive  $\delta$  and  $k \geq 0$ . Then  $\omega = \omega'$ .

There is the action of the automorphism group of  $X$  on Ricci-flat Kähler metrics, in this sense, the uniqueness theorem does not hold on non-compact Kähler manifolds. However if we impose the following boundary condition on metrics, we obtain the uniqueness theorem of Ricci-flat conical Kähler metrics in each Kähler class

**Theorem 1.8** Let  $\omega$  and  $\omega'$  be two Ricci-flat conical Kähler metrics on  $X$  of dimension  $n$  with  $[\omega] = [\omega'] \in H^2(X)$ . We assume that  $X$  has a cylindrical boundary  $C(S) = \mathbb{R} \times S$  with  $H^1(S) = \{0\}$ . If  $\|e^{\lambda t}(\omega - \omega')\|_{C^k} < \infty$ , for a constant  $\lambda > n$ , then  $\omega = \omega'$ .

## 2 Proof of the existence theorem

### 2.1 The Laplacian on conical Riemannian manifolds

Let  $(X, \bar{g})$  be a  $2n$  dimensional cylindrical Riemannian manifold with cylindrical boundary  $C(S)$  and  $g_{\text{cyl}}$  the cylinder metric on  $C(S) = \mathbb{R} \times S$  as in section 1,  $g_{\text{cyl}} = dt^2 + g_S$ , where  $t \in \mathbb{R}$  is the cylinder parameter on  $C(S)$  and  $g_S$  is a Riemannian metric on the manifold  $S$  of dimension  $2n - 1$ . We denote by  $\|f\|_{L_k^p}$  the Sobolev norm of a function  $f$  on  $X$  with respect to the cylindrical metric  $\bar{g}$ ,

$$\|f\|_{L_k^p} = \sum_{i=0}^k \left( \int_X |\nabla^i f|^p \text{vol}_{\text{cyl}} \right)^{\frac{1}{p}},$$

where  $\text{vol}_{\text{cyl}}$  is the volume form with respect to  $\bar{g}$ . We define a weighted Sobolev norm on  $X$  by using the exponential function  $e^{\delta t}$ ,

$$\|f\|_{L_{k,\delta}^p} = \|e^{\delta t} f\|_{L_k^p} \quad (2.1)$$

The weighted Sobolev space is the completion of  $C^\infty$  functions on  $X$  with compact support with respect to the weighted Sobolev norm. Note that a cylindrical Riemannian manifold is complete. We also define a weighted Hölder norm as  $\|f\|_{C_\delta^{k,\alpha}} = \|e^{\delta t} f\|_{C^{k,\alpha}}$  for  $k$  and  $0 < \alpha < 1$  with respect to the cylindrical metric. Recall that a Riemannian metric  $g$  is conical if  $r^{-2}g = e^{-2t}g$  is a cylindrical metric on  $X$ . On a conical Riemannian manifold  $(X, g)$ , we use the weighted Sobolev norm  $\|f\|_{L_{k,\delta}^p}$  with respect to the cylindrical Hermitian metric  $r^{-2}g$  and the weighted Hölder norm is also defined in terms of the cylindrical metric  $r^{-2}g$ . If a function  $f$  lies in  $C_\delta^{k,\alpha}$  implies that  $f$  decays exponentially with order  $O(e^{-\delta t})$  together with its derivatives. (We call  $\delta$  the weight.) Then the Laplacian  $\Delta_g$  gives the bounded linear operator from  $L_{k+2,\delta}^p$  to  $L_{k,\delta+2}^p$ .

**Remark 2.1** Let  $\text{vol}_{\text{cone}}$  be the volume form with respect to the conical metric  $g$ . Then note that

$$\text{vol}_{\text{cone}} = r^{-2n} \text{vol}_{\text{cyl}}.$$

Let  $*_g$  be the Hodge star operator with respect to the conical metric  $g$  and  $*_{\text{cyl}}$  the Hodge star operator with respect to the cylindrical metric  $r^{-2}g$ .

**Lemma 2.2** We define a differential operator  $P$  by

$$P = r^2 \Delta_g.$$

Then the operator  $P$  is given by

$$P = r^2 \Delta_g = \Delta_{\text{cyl}} - (2n - 2) \frac{\partial}{\partial t} \quad (2.2)$$

on the cylinder boundary  $C(S) = X \setminus K \cong \mathbb{R} \times S$ , where  $t$  is the cylinder parameter and  $\Delta_{\text{cyl}}$  the Laplacian with respect to the cylinder metric  $r^{-2}g$ .

*proof* Since  $\Delta_g = *_g d *_g d$  and  $\Delta_{\text{cyl}} = *_{\text{cyl}} d *_{\text{cyl}} d$ , comparing two Hodge star operators  $*_g$  and  $*_{\text{cyl}}$ , we have (2.2). q.e.d.

The cylinder metric  $g_{\text{cyl}} = dt^2 + g_S$  on  $C(S)$  gives the Laplacian  $\Delta_{g_{\text{cyl}}} = (-i \frac{\partial}{\partial t})^2 + \Delta_S$  which is invariant under the translation of cylinder parameter  $t$ , where  $\Delta_S$  is the Laplacian of  $(S, g_S)$ .

Let  $I(P)$  be the translation-invariant operator given by

$$I(P) = \Delta_{g_{\text{cyl}}} - (2n - 2) \frac{\partial}{\partial t} \quad (2.3)$$

$$= (-i \frac{\partial}{\partial t})^2 + \Delta_S - (2n - 2) \frac{\partial}{\partial t}. \quad (2.4)$$

Then the all coefficients of the operators  $P - I(P)$  decays exponentially together with their derivatives, which implies that we can apply the theory of elliptic differential operators on cylindrical Riemannian manifolds developed by [18], [19]. (Note that these operators are called  $b$ -operators in [19].) We can select suitable weighted Sobolev spaces to obtain the operator  $P$  as the Fredholm operator. Substituting  $\lambda$  for  $-i \frac{\partial}{\partial t}$  into (2.4), we define the family of the operators  $I(P, \lambda)$  on  $S$  given by

$$I(P, \lambda) = \lambda^2 - (2n - 2)\lambda i + \Delta_S$$

which is called the indicial family parametrized by  $\lambda \in \mathbb{C}$ . The operator

$$I(P, \lambda) : L_{k+2}^p(S) \rightarrow L_k^p(S)$$

is an isomorphism for all  $\lambda \in \mathbb{C} \setminus \text{Spec}(P)$ , that is, The operator  $I(P)$  does not admit a bounded inverse for  $\lambda \in \text{Spec}(P)$ . The properties of the set  $\text{Spec}(P)$  are developed in [18] and [19]. In our cases,  $\text{Spec}(P)$  is explicitly described as

**Lemma 2.3**

$$\text{Spec}(P) = \{ 0, (2n - 2)\sqrt{-1}, \mu_j^+ \sqrt{-1}, \mu_j^- \sqrt{-1} \mid j = 1, 2, \dots \}$$

where  $\mu_j^\pm \sqrt{-1}$  are two solutions of the quadratic equation  $x^2 - (2n - 2)\sqrt{-1}x + \lambda_j = 0$  and  $\lambda_j$  is the  $j$ -th eigenvalue of the Laplacian  $\Delta_S$  on  $S$ .

*proof* It follows from  $I(P, \lambda) = \lambda^2 - (2n - 2)\lambda i + \Delta_S$ . q.e.d.

Let  $\text{ImSpec}(P)$  be the set consisting of the imaginary parts of elements of  $\text{Spec}(P)$ . Then  $\text{ImSpec}(P)$  is given by

$$\cdots < \mu_2^- < \mu_1^- < 0 < 2n - 2 < \mu_1^+ < \mu_2^+$$

We select the weighted Sobolev space  $L_{k,\delta}^p$  with weight  $\delta \notin \text{ImSpec}(P)$ . Then from theorem 1.1 in [RMc] and [Mel],  $P : L_{k+2,\delta}^p \rightarrow L_{k,\delta}^p$  is a Fredholm operator with index  $\text{ind}(P, \delta)$ . Further for  $0 < \delta < n - 1$ , since the Laplacian  $\Delta_g$  is self-adjoint, applying the theorem 7.4 (see page 436) in [18], we obtain

$$\text{ind}(P, \delta + n - 1) + \text{ind}(P, -\delta + n - 1) = 0$$

There is the formula for the index of the cylindrical operators which depend on weights  $\delta$  (see the theorem 1.2 in [18]), Applying the theorem 1.2, (Note that  $N(-\delta + n - 1, \delta + n - 1) = 0$ ), we have

$$-\text{ind}(P, \delta + n - 1) + \text{ind}(P, -\delta + n - 1) = 0.$$

Thus we obtain  $\text{ind}(P, \delta) = 0$  for  $0 < \delta < 2(n - 1)$ .

By the result, we have the solvability of the equation of the Laplacian  $\Delta_g$ .

**Proposition 2.4** *Let  $(X, g)$  be a  $2n$ -dimensional conical Riemannian manifold and  $\Delta_g$  the Laplacian with respect to the conical metric  $g$ . For a weight  $\delta$  with  $0 < \delta < 2n - 2$ , the Laplacian  $\Delta_g$  gives an isomorphism between  $L_{k+2,\delta}^p$  and  $L_{k,\delta+2}^p$ , for positive integers  $p, k$  and  $0 < \alpha < 1$  with  $\frac{1}{p} < \frac{k-\alpha}{2n}$ .*

*proof* It follows from the Sobolev embedding theorem that  $L_{k+2,\delta}^p \subset C_\delta^{2,\alpha}$ . By the Maximum principle, a harmonic function attains its Maximum and Minimum at the boundary. Since a function in  $C_\delta^{2,\alpha}$  decays exponentially, we have  $\ker \Delta_g = \{0\}$  which implies that  $\ker P = \ker \Delta_g = \{0\}$ . Since  $\text{ind}(P, \delta) = 0$ , we have an isomorphism  $P : L_{k+2,\delta}^p \rightarrow L_{k,\delta}^p$ . Since  $\Delta_g$  is the composition  $r^{-2}P$ ,  $\Delta_g : L_{k+2,\delta}^p \rightarrow L_{k,\delta+2}^p$  is an isomorphism also. q.e.d.

Then we have the followings,

**Proposition 2.5** *Let  $(X, g)$  be a  $2n$ -dimensional conical Riemannian manifold and  $\Delta_g$  the Laplacian with respect to the conical metric  $g$ . For a weight  $\delta$  with  $0 < \delta < 2n - 2$ , the Laplacian  $\Delta_g$  gives an isomorphism between  $C_\delta^{k+2,\alpha}$  and  $C_{\delta+2}^{k,\alpha}$ . In other words, there is a unique solution  $u \in C_\delta^{k+2,\alpha}$  of the equation  $\Delta_g u = f$  for all  $f \in C_{\delta+2}^{k,\alpha}$ , where  $k \geq 0$ .*

**Remark 2.6** *If  $(X, g)$  is an ALE space, then the proposition 2.5 implies the theorem 8.3.5 (a) in [15]. The Hölder space  $C_\beta^{k, \alpha}$  in [15] coincides with our Hölder space  $C_\delta^{k, \alpha}$  with  $\delta = -\beta$ .*

We need the following lemma for a proof of the proposition 2.5.

**Lemma 2.7** *Let  $(X, g)$  be a conical Riemannian manifold with the cylinder parameter  $t$  and  $\Delta_g$  the Laplacian with respect to the conical metric  $g$ . We take a smaller weight  $\tilde{\delta}$  with  $0 < \tilde{\delta} < \delta < 2n - 2$ . We assume that a function  $u \in C_{\tilde{\delta}}^k$  satisfies*

$$\Delta_g u = h,$$

*and we already have a bound of  $C^0$ -norm  $\|u\|_{C^0} < C_1$  and a weighted  $C^0$ -norm  $\|e^{(2+\delta)t}h\|_{C^0} < C_2$ . Then there is a constant  $C > 0$  depending only on  $C_1, C_2$  and  $g$  such that*

$$\|e^{\delta t}u(x)\|_{C^0} < C$$

*proof of lemma 2.7* Let  $\Delta_{\text{cone}}$  be the Laplacian with respect to the cone metric  $g_{\text{cone}}$  on the cone  $C(S)$ . Then we see that

$$\Delta_{\text{cone}} = e^{-2t} \left( -\left(\frac{\partial}{\partial t}\right)^2 - (2n-2)\frac{\partial}{\partial t} + \Delta_S \right)$$

Thus we have

$$\Delta_{\text{cone}} e^{-\delta t} = \delta(2n-2-\delta)e^{-(2+\delta)t} > 0$$

on the cone  $C(S)$ , for  $0 < \delta < 2n-2$ . Then from the definition of the conical metric, there are constants  $C_3 > 0$  and  $T > 0$  such that

$$\Delta_g e^{-\delta t} \geq C_3 e^{-(2+\delta)t}, \quad (2.5)$$

on the region  $D_T := \{t > T\}$ . We take a constant  $C$  satisfying the two inequalities,

$$C_3 C > C_2 \quad (2.6)$$

$$C e^{-\delta T} - C_1 > 0. \quad (2.7)$$

Then from (2.5) and (2.6), we have

$$\Delta_g (C e^{-\delta t} \pm u) > C C_3 e^{-(2+\delta)t} \pm h \quad (2.8)$$

$$> C C_3 e^{-(2+\delta)t} - C_2 e^{-(2+\delta)t} \quad (2.9)$$

$$> 0 \quad (2.10)$$

From (2.7), we also have an inequality on the compact set  $\{t \leq T\}$

$$(Ce^{-\delta t} \pm u)(x) \geq Ce^{-\delta T} - C_1 \geq 0, \quad (2.11)$$

for all  $x \in \{t \leq T\}$ . If the function  $Ce^{-\delta t} \pm u$  have its Minimum at  $x_0 \in D_T$ , then  $\Delta_g(Ce^{-\delta t} \pm u)(x_0) \leq 0$ . Thus from (2.10),  $Ce^{-\delta t} \pm u$  can not have the Minimum on  $D_T$ . Since  $\lim_{t \rightarrow \infty} u(x) = 0$ , It follows from (2.11) that  $(Ce^{-\delta t} \pm u)(x) \geq 0$ , for all  $x \in X$ . Hence we have  $\|e^{\delta t} u\|_{C^0} < C$ . q.e.d.

*proof of proposition 2.5* Since  $u \in C_\delta^{k+2,\alpha}$  decays exponentially, it follows that  $\ker \Delta_g = \{0\}$ . For a smaller weight  $0 < \tilde{\delta} < \delta$ , any function  $f \in C_{2+\tilde{\delta}}^{k,\alpha}$  is a function in  $L_{k,\tilde{\delta}+2}^p$ . Then it follows from the proposition 2.4 that there is a function  $u \in L_{k+2,\tilde{\delta}}^p$  such that  $\Delta_g u = f$ . Since  $u$  is a function in  $C_{\tilde{\delta}}^{k+2,\alpha}$ , it follows from the lemma 2.7 that  $u \in C_\delta^0$ . We have the following equation,

$$\Delta_g(e^{\delta t} u) = H,$$

where  $H \in C_\delta^{k,\alpha}$  is a function consisting of the terms  $(\Delta_g e^{\delta t})u$ ,  $e^{\delta t} f$  and  $(\nabla_g e^{\delta t})(\nabla_g u)$ . Thus from the Schauder estimate, we have  $e^{\delta t} u \in C^{k+2,\alpha}$ . Hence we have a unique solution  $u \in C_\delta^{k+2,\alpha}$  of the equation  $\Delta_g u = f$ , for  $f \in C_\delta^{k+2,\alpha}$ . q.e.d.

## 2.2 The Sobolev inequality for conical Riemannian manifolds

**Proposition 2.8** *Let  $(X, g)$  be a conical Riemannian manifold of dimension  $2n$ . Then every function  $f \in L^{2\varepsilon} \cap L_1^2$  satisfies:*

$$\|f\|_{L^{2\varepsilon}} < S \|df\|_{L^2},$$

where  $\varepsilon = \frac{n}{n-1}$  and  $S$  is a constant which depends only on  $\alpha$ ,  $n$  and  $g$ . Note that the point-wise norm and  $L^2$ -norm are defined by the cone metric  $g$  in this proposition.

Note that the curvature of a conical Riemannian manifold is bounded. In the proposition, we do not assume the condition:

$$\int_X f \text{vol} = 0,$$

which is necessary for the Sobolev inequality for compact Riemannian manifolds. The theorem is already known for geometers, however for the sake of readers, the author gives a proof. Our proof relies on a well-known result on compact Riemannian manifolds :

**Theorem 2.9** [11] *Let  $M$  be a  $2n$  dimensional compact Riemannian manifold on which the Ricci curvature  $\text{Ric}(g)$  satisfies*

$$\text{diam}(X, g)^2 \text{Ric}(g) \geq -\alpha g$$

*for a constant  $\alpha$ . Then there is a constant  $\kappa(2n, \alpha)$  depending only on  $2n$  and constant  $\alpha$  such that for all  $C^\infty$  function  $f$  on  $X$  with  $\int_X f \text{vol} = 0$ , the following Sobolev inequality holds,*

$$\kappa(2n, \alpha) \frac{\text{Vol}(M, g)^{\frac{1}{2n}}}{\text{diam}(X, g)} \|f\|_{L^{2\varepsilon}} \leq \|df\|_{L^2}$$

*proof* (theorem 2.8) We extend the cylinder parameter  $t$  as a  $C^\infty$  function on  $X$  and define a compact subset  $X_T$  by

$$X_T = \{x \in X \mid t(x) \leq T\},$$

for  $T > 0$ . Taking two copies  $X_T^1, X_T^2$  of  $X_T$ , we obtain a compact Riemannian manifold  $M_T$  to glue  $X_T^1$  and  $X_T^2$  at the boundary  $\partial X_T = \{x \in X \mid t(x) = T\}$  since  $X$  is a conical Riemannian manifold, the volume  $\text{vol}(M_T)$  and the diameter  $\text{diam}(M_T)$  of  $M_T$  satisfy

$$\lim_{T \rightarrow \infty} \frac{\text{diam}(M_T)}{\text{Vol}(M_T)^{\frac{1}{2n}}} < C$$

for a constant  $C$  which does not depend on  $T$ . For a  $C^\infty$  function with compact support, we define a function  $f_T$  on  $M_T$  by

$$f_T = \begin{cases} +f(x), & x \in M_T^1 \\ -f(x), & x \in M_T^2 \end{cases}$$

Then  $\int_{M_T} f_T \text{vol} = 0$ , we apply the theorem 2.9 to the function  $f_T$  and have

$$\|f_T\|_{L^{2\varepsilon}} < \kappa(2n, \alpha)^{-1} \|df_T\|_{L^2}$$

Since we have

$$\lim_{T \rightarrow \infty} \|f_T\|_{L^{2\varepsilon}} = 2\|f\|_{L^{2\varepsilon}}, \quad (2.12)$$

$$\lim_{T \rightarrow \infty} \|df_T\|_{L^2} = 2\|df\|_{L^2} \quad (2.13)$$

we have the inequality

$$\|f\|_{L^{2\varepsilon}} \leq \kappa(2n, \alpha)^{-1} \|df\|_{L^2},$$

for a  $C^\infty$  function  $f$  with compact support. Since  $(X, g)$  is complete, we obtain the result. q.e.d.

### 2.3 The inequality of solutions of the Monge-Ampère equation

Let  $(X, g, \omega)$  be a conical Kähler manifold. We use the same notation as in the theorem 1.5. We assume that there is a  $C^\infty$  function  $u$  on  $X$  which satisfies the equation

$$\omega_u^n = F\omega^n,$$

where

$$\omega_u = \omega + dd^c u$$

is a Kähler form on  $X$  and a  $C^\infty$  positive function  $F$  satisfies  $F - 1 \in C_{2+\delta}^{k,\alpha}$  and  $u \in C_\delta^{k+2,\alpha}$  and  $d^c = \frac{1}{2\sqrt{-1}}(\partial - \bar{\partial})$ . We select a weight  $\delta$  with  $0 < \delta < 2n - 2$  and a natural number  $p$  with  $\delta p + 2 > 2n$ . Then since  $\omega^n = e^{2nt} \text{vol}_{\text{cyl}}$ , the function  $u|u|^{p-2}(1 - F)$  is integrable with respect to the volume form  $\omega^n$ . Thus we have

$$\begin{aligned} \int_X u|u|^{p-2}(1 - F)\omega^n &= \int_X u|u|^{p-2}(\omega^n - \omega_u^n) \\ &= \int_X u|u|^{p-2}(-dd^c u) \wedge (\omega^{n-1} + \dots + \omega_u^{n-1}) \end{aligned}$$

Since  $\delta p + 2 > 2n$  and  $u|u|^{p-2}d^c u \wedge (\omega^{n-1} + \dots + \omega_u^{n-1})$  decays exponentially with respect to the cylinder parameter  $t$ , we can apply the Stokes theorem,

$$\int d(u|u|^{p-2}d^c u \wedge (\omega^{n-1} + \dots + \omega_u^{n-1})) = 0$$

Substituting  $d(u|u|^{p-2}) = (p-1)|u|^{p-2}du$ , we obtain

$$\int_X u|u|^{p-2}(1 - F)\omega^n = (p-1) \int_X |u|^{p-2}du \wedge d^c u \wedge (\omega^{n-1} + \dots + \omega_u^{n-1}) \quad (2.14)$$

Since  $du \wedge d^c u \wedge (\omega^i \wedge \omega_u^{n-1-i}) \geq 0$  at each point on  $X$ , we have

$$du \wedge d^c u \wedge (\omega^{n-1} + \dots + \omega_u^{n-1}) \geq du \wedge d^c u \wedge \omega^n$$

Substituting it into (2.14), we have

$$\int_X u|u|^{p-2}(1 - F)\omega^n \geq (p-1) \int_X |u|^{p-2}du \wedge d^c u \wedge \omega^{n-1} \quad (2.15)$$

$$= \frac{p-1}{n} \int_X |u|^{p-2} |du|_g^2 \omega^n \quad (2.16)$$

$$= \frac{4(p-1)}{p^2 n} \int_X |(d|u|^{\frac{p}{2}})|_g^2 \omega^n \quad (2.17)$$

where we are using

$$\frac{1}{4}p^2|u|^{p-2}|du|_g^2 = |(d|u|^{\frac{p}{2}})|_g^2$$

and  $|(d|u|^{\frac{p}{2}})|_g^2$  is the norm of the 1-form  $d|u|^{\frac{p}{2}}$  with respect to the conical metric  $g$ . Hence we obtain the inequality,

**Proposition 2.10**

$$\int_X |(d|u|^{\frac{p}{2}})|_g^2 \omega^n \leq \frac{p^2 n}{4(p-1)} \int_X u|u|^{p-2}(1-F)\omega^n$$

For simplicity, we denote by  $Kp$  the constant  $\frac{p^2 n}{4(p-1)}$ .

## 2.4 An inequality for the induction

**Lemma 2.11** *Let  $A(p)$  be a positive function of one variable  $p \in \mathbb{R}$  which satisfies the following inequality,*

$$A(p\varepsilon)^p \leq c_1 p (A(p-1))^{p-1} \quad (2.18)$$

where  $c_1$  is a constant and  $\varepsilon = \frac{n}{n-1} > 1$ . We assume that there is a natural number  $N$  such that  $A(p) < c_0$  for all  $p \in [N, N\varepsilon]$ . Then  $A(p)$  satisfies the following inequality

$$A(p) \leq s_1 (s_2 p)^{-\frac{s_3}{p}}, \quad (2.19)$$

where constants  $s_1, s_2$  depend only on  $c_0, c_1$  and  $N$  and  $s_3 = \frac{2\varepsilon}{\varepsilon-1}$ . In particular, we have

$$\lim_{p \rightarrow \infty} A(p) = s_1$$

which implies that  $A(p)$  is bounded by a constant which does not depend on  $p$ .

*proof* We take a constant  $s_2$  which satisfies

$$2^{s_3} c_1 p^{-1} \leq (\varepsilon)^{-2(n-1)} s_2^2, \quad (2.20)$$

for all  $p > N$ . (It is possible because the left hand side is bounded.) and then we choose a constant  $s_1 > 1$  with

$$c_0 \leq s_1 (s_2 p)^{-\frac{s_3}{p}}, \quad (2.21)$$

for all  $p$  (It is also possible since  $\lim_{p \rightarrow \infty} (s_2 p)^{-\frac{s_3}{p}} = 1$ .) Then we shall show that  $A(p)$  satisfies the inequality (2.19) by the induction on  $m$ . The inequality

(2.19) holds for  $p \in [N, N\varepsilon]$ . We assume that  $A(p)$  satisfies the inequality (2.19) for all  $p \in [N, N\varepsilon^m]$ . Then we shall show that the inequality holds for  $p \in [N, N\varepsilon^{m+1}]$ . From our assumption,  $A(p)$  and  $A(p-1)$  satisfies the inequality (2.19) and then applying (2.18), we have

$$A(p\varepsilon)^p \leq c_1 p s_1^{p-1} (s_2(p-1))^{-s_3} \leq (2^{s_3} c_1 p) s_1^p (s_2 p)^{-s_3} \quad (2.22)$$

by using  $(\frac{p}{p-1})^{s_3} < 2^{s_3}$ . If we have the following inequality

$$(2^{s_3} c_1 p) s_1^p (s_2 p)^{-s_3} \leq s_1^p (s_2 p \varepsilon)^{-\frac{s_3}{\varepsilon}}, \quad (2.23)$$

then  $A(p\varepsilon)$  satisfies the inequality (2.19). Thus it suffices to show (2.23). The inequality (2.23) is equivalent to

$$(2^{s_3} c_1 p) p^{-s_3 + \frac{s_3}{\varepsilon}} \leq (\varepsilon)^{\frac{-s_3}{\varepsilon}} s_2^{-\frac{s_3}{n}} \quad (2.24)$$

Since  $-s_3 + \frac{s_3}{\varepsilon} = -\frac{s_3}{n} = -2$  and  $\frac{s_3}{\varepsilon} = 2(n-1)$ , (2.24) is

$$2^{s_3} c_1 p^{-1} \leq (\varepsilon)^{-2(n-1)} s_2^2 \quad (2.25)$$

This is the inequality which  $s_2$  satisfies. Thus the result follows from the induction. q.e.d.

## 2.5 The openness

Our proof of the theorem relies on the continuity method. We consider a family of the Monge-Ampère equation parametrised by  $s \in [0, 1]$ . For  $F_s = 1 - s + sF$ , we define

$$\omega_{u_s}^n = F_s \omega^n \quad (2.26)$$

Let  $S$  be a subset of  $[0, 1]$  defined by

$$S = \{ s \in [0, 1] \mid \text{the equation (2.26) has a solution } u_s \in C_\delta^{k+2, \alpha} \},$$

Note that  $F - 1 \in C_\delta^{k, \alpha}$ , for all  $\tilde{\delta} \leq \delta$ . Since  $F_0 = 1$ , the equation (2.26) has the trivial solution 0 for  $s = 0$ . If we show that  $S$  is open and closed, then there is a solution  $u_1$  for  $s = 1$  which gives the solution of the Monge-Ampère equation in the theorem. In order to show that  $S$  is open, taking the derivative of the equation (2.26) at  $u_s$ , we have the equation of the Laplacian

$$-\Psi_s \Delta_s u_s = \dot{F}_s, \quad (2.27)$$

where  $u_s = \frac{d}{ds} u_s$  and  $\dot{F}_s = F - 1$  and  $\Psi_s$  is a bounded function given by

$$\Psi_s = \frac{1}{2} \frac{\omega_{u_s}^n}{\omega^n}$$

and  $\Delta_s$  is the Laplacian with respect to the Kähler metric  $\omega_{u_s}$ . Since  $u_s \in C_\delta^{k+2,\alpha}$ , the metric  $\omega_{u_s}$  is a conical Kähler metric. Then since  $F-1 \in C_{2+\delta}^{k,\alpha}$ , it follows from the proposition 2.5 there is a unique solution  $u_s$  of the equation (2.27). We define two Banach manifolds  $\mathcal{M} = \{u \in C_\delta^{k+2,\alpha} \mid \omega_u > 0\}$ .  $\mathcal{N} = \{f+1 \mid f \in C_{2+\delta}^{k,\alpha}\}$ . Then the smooth map  $\Psi : \mathcal{M} \rightarrow \mathcal{N}$  is defined by

$$\Psi(u) = \frac{\omega_u^n}{\omega^n}.$$

The differential of  $\Psi$  at  $u$  is the map  $d\Psi_u : C_\delta^{k+2,\alpha} \rightarrow C_\delta^{k,\alpha}$  which is given by

$$d\Psi_u(\dot{u}) = -\Psi_u \Delta_u,$$

where  $\Delta_u$  is the Laplacian with respect to the Kähler metric  $\omega_u$ . Then from the proposition 2.5,  $d\Psi_u$  is isomorphism and it follows from the implicit function theorem that  $\Psi$  is locally an isomorphism. Then  $S$  is an open set.

## 2.6 $C^0$ -estimates

We shall show the  $C^0$ -estimate of the Monge-Ampère equation  $\omega_{u_s}^n = F_s \omega^n$ . For simplicity, we write  $u$  for  $u_s$  and  $F$  for  $F_s$ . We use  $\omega^n$  as the (conical) volume form of the integrant in this subsection. Recall that  $u \in C_\delta^{k+2,\alpha}$  and  $F-1 \in C_\delta^{k,\alpha}$ , for  $0 < \delta < 2n-2$ . Substituting  $f = |u|^{\frac{p}{2}}$  into the Sobolev inequality (proposition 2.8), we have

$$\left( \int_X |u|^{p\varepsilon} \omega^n \right)^{\frac{1}{\varepsilon}} \leq S^2 \int_X |d|u|^{\frac{p}{2}}|_g^2 \omega^n$$

Applying the inequality (proposition 2.10), we have

$$\left( \int_X |u|^{p\varepsilon} \omega^n \right)^{\frac{1}{\varepsilon}} \leq S^2 K p \int_X u |u|^{p-2} (1-F) \omega^n \quad (2.28)$$

We set  $A(p)$  by

$$A(p) = \left( \int_X |u|^p \omega^n \right)^{\frac{1}{p}}$$

and  $c_1 = S^2 K \|1-F\|_{C^0}$  by the  $C^0$ -norm of  $1-F$ . Then the inequality (2.28) implies

$$A(p\varepsilon)^p \leq c_1 (A(p-1))^{p-1}. \quad (2.29)$$

Then we shall show an estimate of the  $C^0$ -norm by using the lemma 2.11. Recall  $p\delta > 2n-2$  in the proposition 2.10. We set  $r > 1$  by  $(p-1)r = p\varepsilon$  and define  $s > 1$  by  $\frac{1}{s} + \frac{1}{r} = 1$ . Then it follows

$$(\delta+2)s > 2n. \quad (2.30)$$

Applying the Hölder inequality to the right hand side of (2.28) with the exponents  $r, s$ , we have

$$A(p\varepsilon)^p \leq S^2 K p (A(p\varepsilon))^{\frac{p\varepsilon}{r}} \left( \int_X |1 - F|^s \omega^n \right)^{\frac{1}{s}} \quad (2.31)$$

Since  $p - \frac{p\varepsilon}{r} = 1$ , we have

$$A(p\varepsilon) \leq S^2 K p \left( \int_X |1 - F|^s \omega^n \right)^{\frac{1}{s}}.$$

Note that from (2.30),  $(\int_X |1 - F|^s \omega^n) < \infty$ . Thus for  $p\delta > 2n - 2$ ,  $A(p)$  is bounded. Then together with (2.29), we can apply the lemma 2.11 to obtain

$$A(p) = \left( \int_X |u|^p \omega^n \right)^{\frac{1}{p}} < C,$$

where  $C$  does not depend on  $p$ . Thus we obtain an estimate of  $C^0$ -norm of  $u$ .

## 2.7 $C^2$ -estimate

There are many good references for an estimate of  $C^2$ -norm of a solution  $u$  of the Monge-Ampère equation [1], [15], [21], [24]. Bando-Kobayashi [3] developed a way for the  $C^2$ -estimate to apply the Schwartz lemma which makes a comparison of two metrics  $\omega$  and  $\omega_u$  by using their curvature, which is clear from a geometric point of view.

We shall give a quick view of the  $C^2$ -estimate on our conical Kähler manifolds. Let  $\text{tr}_{\omega_u} \omega$  be the trace of  $\omega$  by the Kähler metric  $\omega_u$ . It follows that  $\text{tr}_{\omega_u} \omega > 0$  is the norm of  $\omega$  by  $\omega_u$  which gives the  $C^2$ -norm of  $u$ . According to the Bando-Kobayashi's result, there is a constant  $C$  depending only on the curvature of  $\omega$  and the Ricci curvature of  $\omega_u$  such that the following inequality holds,

$$-\square_u \log \text{tr}_{\omega_u} \omega \geq -C(1 + \text{tr}_{\omega_u} \omega)$$

where  $\square_u$  is the complex Laplacian with respect to the Kähler metric  $\omega_u$ . Since  $u$  is a solution of the Monge-Ampère equation, the Ricci curvature of  $\omega_u$  is already bounded. Then by using,

$$-\square_u u = n - \text{tr}_{\omega_u} \omega$$

we have

$$-\square_u (\log \text{tr}_{\omega_u} \omega - (C + 1)u) \geq \text{tr}_{\omega_u} \omega - (C + 1)n - C$$

Since the function  $u$  decays exponentially with order  $O(e^{-\delta t})$ , the function  $(\log \operatorname{tr}_{\omega_u} \omega - (C+1)u)$  is maximum at a point  $x_0 \in X$ . Then

$$0 \geq -\square_u(\log \operatorname{tr}_{\omega_u} \omega - (C+1)u)(x_0) \quad (2.32)$$

$$\geq (\operatorname{tr}_{\omega_u} \omega - (C+1)n - C)(x_0) \quad (2.33)$$

Hence

$$(C+1)n + C \geq \operatorname{tr}_{\omega_u} \omega(x_0)$$

It follows that

$$(\log \operatorname{tr}_{\omega_u} \omega - (C+1)u)(x_0) \geq (\log \operatorname{tr}_{\omega_u} \omega - (C+1)u)(x), \quad \forall x \in X$$

Thus we have the upper bound for  $\operatorname{tr}_{\omega_u} \omega$ ,

$$\sup_X \log \operatorname{tr}_{\omega_u} \omega < \log((C+1)n + C) + 2(C+1)\|u\|_{C^0}.$$

By using the Monge-Ampère equation  $\omega_u^n = F\omega^n$ , we have  $C^2$ -estimate of  $u$  and we obtain an apriori constant  $C > 0$  such that

$$C^{-1}\omega < \omega_u < C\omega.$$

## 2.8 $C_{\tilde{\delta}}^0$ -estimate

In this subsection, we shall give an estimate of the weighted  $C^0$ -norm  $u$  of a solution of the Monge-Ampère equation. In the process of our estimate, we need to choose a smaller weight  $\tilde{\delta}$  with  $0 < \tilde{\delta} < \delta$ . We can obtain a way of the weighted  $C^0$ -estimate by applying the method by Joyce, (see section 8 in [15]). Our notation are the same as in the previous subsection.

At first we have the following weighted inequality,

**Lemma 2.12** *For a constant  $q > 0$  satisfying*

$$q < p\delta - 2n + 2$$

*we have*

$$\begin{aligned} \int_X r^q |(d|u|^{\frac{p}{2}})|_g^2 \omega^n &\leq \frac{p^2 m}{4(p-1)} \int_X r^q u |u|^{p-2} (1-F) \omega^n \\ &\quad + q^2 C \int_X r^{q-2} |u|^p \omega^n \end{aligned}$$

*where  $r = e^t$  and a constant  $C$  depends only on the  $C^2$ -norm of  $u$  given in the previous section.*

*proof* The condition

$$q < p\delta - 2n + 2$$

implies that the Stokes theorem holds,

$$\int_X d(r^q u |u|^{p-2} d^c u \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1})) = 0.$$

Then the result follows as in the way of the proposition 2.10 q.e.d.

The following lemma is a slight generalization of the lemma 2.11

**Lemma 2.13** *Let  $B(p)$  be a positive function with one variable  $p \in \mathbb{R}_{>0}$ . We assume that  $B(p)$  satisfies the following,*

$$B(p\varepsilon)^p \leq c_1 p B(p-1)^{p-1} + (c_2 p^2 + c_3) B(p)^p \quad (2.34)$$

where  $\varepsilon = \frac{n-1}{n}$  and there are a natural number  $N$  and a positive  $c_0$  such that  $B(p) < c_0$  for all  $p \in [N, N\varepsilon]$ . Then there are constants  $s_1, s_2$  depending only on  $c_0, c_1, c_2, c_3$  such that for all  $p > N$ ,  $B(p)$  satisfies the inequality,

$$B(p) \leq s_1 (s_2 p)^{-\frac{s_3}{p}}$$

where  $s_3 = \frac{2\varepsilon}{\varepsilon-1}$ . In particular, we have

$$\lim_{p \rightarrow \infty} B(p) = s_1$$

Thus  $B(p)$  is bounded by a constant which does not depend on  $p$ .

*proof* The proof is essentially same as the one in lemma 2.11. q.e.d.

We shall start our  $C_{\tilde{\delta}}^0$ -estimate, paying our attention on the weight. We choose a weight  $\tilde{\delta}$  satisfying  $0 < \tilde{\delta} < \delta$ . and try to obtain a  $C_{\tilde{\delta}}^0$ -norm of  $u$ . We set

$$q = p\tilde{\delta} - 2n + 2$$

Then  $q$  satisfies

$$-p\delta + 2n - 2 + q < 0 \quad (2.35)$$

which is the condition of the lemma 2.12. We define  $B(p)$  by the weighted  $L^p$ -norm of  $u$  with respect to the cylindrical metric.

$$B(p) = \|u\|_{L_{0,\delta}^p} = \left( \int_X |e^{\tilde{\delta}t} u|^p \text{vol}_{\text{cyl}} \right)^{\frac{1}{p}}$$

Note  $\omega^n = e^{2nt} \text{vol}_{\text{cyl}}$ . Then we have

$$\int_X e^{\varepsilon qt} |u|^{p\varepsilon} \omega^n = \int_X e^{\varepsilon p \tilde{\delta} t} |u|^{p\varepsilon} e^{(-2n+2)\varepsilon t} e^{2nt} \text{vol}_{\text{cyl}} \quad (2.36)$$

Since  $\varepsilon = \frac{n}{n-1}$ , we have  $(-2n+2)\varepsilon + 2n = 2(1-n)\frac{n}{n-1} + 2n = 0$ . Then it follows

$$\int_X e^{\varepsilon q t} |u|^{p\varepsilon} \omega^n = \int_X |e^{\tilde{\delta} t} u|^{p\varepsilon} \text{vol}_{\text{cyl}} = B(p\varepsilon)^{p\varepsilon} \quad (2.37)$$

We substitute  $e^{\frac{q}{2}t} |u|^{\frac{p}{2}}$  into the Sobolev inequality in the proposition 2.8. Since in the Sobolev inequality, the norm is given by the conical metric  $\omega$  and the volume form is also  $\omega^n$ , the left hand side of the inequality is

$$\|e^{\frac{q}{2}t} |u|^{\frac{p}{2}}\|_{L^{2\varepsilon}}^2 = \left( \int_X e^{q\varepsilon t} |u|^{p\varepsilon} \omega^n \right)^{\frac{1}{\varepsilon}} = B(p\varepsilon)^p \quad (2.38)$$

and using the norm by the cone metric, we find that the right hand side is

$$\|d\left(e^{\frac{q\tilde{\delta}}{2}t} |u|^{\frac{p}{2}}\right)\|_{L^2}^2 \leq C \int_X e^{q\tilde{\delta}t} |d|u|^{\frac{q}{2}}|^2 \omega^n + Cq\tilde{\delta} \int_X e^{(q-2)\tilde{\delta}t} |u|^p \omega^n \quad (2.39)$$

The second term is given by

$$\int_X e^{(q-2)\tilde{\delta}t} |u|^p \omega^n = \int_X |e^{\tilde{\delta}t} u|^p e^{-2nt} e^{2nt} \text{vol}_{\text{cyl}} = B(p)^p \quad (2.40)$$

Since  $e^{(2+\tilde{\delta})t} |1-F| < C$ , we find that

$$\int_X e^{qt} |u|^{p-1} |1-F| \omega^n = \int_X e^{-\tilde{\delta}t} e^{(q-2)t} |u|^{p-1} e^{(2+\tilde{\delta})t} |1-F| \omega^n \quad (2.41)$$

$$\leq C \int_X e^{-\tilde{\delta}t} e^{p\tilde{\delta}t} |u|^{p-1} \text{vol}_{\text{cyl}} \quad (2.42)$$

$$\leq C \int_X |e^{\tilde{\delta}t} u|^{p-1} \text{vol}_{\text{cyl}} \quad (2.43)$$

$$= CB(p-1)^{p-1} \quad (2.44)$$

Substituting these into the Sobolev inequality and combining the inequality in the proposition 2.12 we obtain

$$\|u_s\|_{L_{0,\tilde{\delta}}^{p\varepsilon}}^p < C_1 p \|u_s\|_{L_{0,\tilde{\delta}}^{p-1}}^{p-1} + (C_2 p^2 + C_2) \|u_s\|_{L_{0,\tilde{\delta}}^p}^p \quad (2.45)$$

$$(2.46)$$

The inequality implies

$$B(p\varepsilon)^p < C_1 p B(p-1)^{p-1} + (C_2 p^2 + C_2) B(p)^p \quad (2.47)$$

We apply a simple trick to show a bound of  $B(p)$  from  $A(p)$  with respect to the weight. Recall

$$A(P) = \left( \int_X |u|^p \omega^n \right)^{\frac{1}{p}} = \left( \int_X |u|^p e^{2nt} \text{vol}_{\text{cyl}} \right)^{\frac{1}{p}}$$

For  $p\delta < 2n - 2$ , we already have a bound  $C$  of  $A(p)$  in the subsection of  $C^0$ -estimate. On the other hand,

$$B(p) = \left( \int_X |u|^p e^{p\tilde{\delta}t} \text{vol}_{\text{cyl}} \right)^{\frac{1}{p}}$$

Thus for  $p$  satisfying  $2n - 2 < p\tilde{\delta} < 2n$ , we have  $B(p) < A(p) < C$ , (where  $\tilde{\delta} < \delta$ .) Applying the lemma 2.13 to our  $B(p)$ , We obtain the estimate  $B(p) < C$ , where  $C$  does not depend on  $p$ . Thus it follows that the  $C^0$ -norm of  $e^{\tilde{\delta}t}u$  is bounded.

## 2.9 $C_\delta^0$ -estimate by the maximum (minimum) principle

We shall obtain an  $C_\delta^0$ -estimate by using the maximum (minimum) principle.

We shall use the equation (2.27) in subsection 2.5 to obtain an  $C_\delta^0$ -estimate of a solution of Monge-Ampère equation,

$$-\Psi_s \Delta_s \dot{u}_s = \dot{F}_s, \quad (2.48)$$

We already have the  $C_\delta^0$ -estimate. Note that  $\Psi_s$  is bounded. Then as in compact Kähler manifolds, we have  $C_\delta^{k,\alpha}$ -estimate, for  $0 < \tilde{\delta} < \delta$ . It implies that we have the  $C_\delta^{k,\alpha}$ -estimate of a solution  $\dot{u}_s$  of the equation (2.27) which does not depend on  $s$ . The difference of every coefficients of the Laplacians  $\Delta_s - \Delta$  decay exponentially with the order  $e^{-(2+\tilde{\delta})t}$ . Then we have constants  $C_3$  and  $T > 0$  such that

$$\Delta_s e^{-\delta t} \geq C_3 e^{-(2+\delta)t},$$

on the region  $\{t > T\}$ . We take a constant  $C$  which satisfies the two inequalities, (2.6) and (2.7), where  $\|\dot{u}_s\|_{C^0} < C_1$  and  $\|e^{(2+\delta)t} \Psi_s^{-1} \dot{F}_s\|_{C^0} < C_2$ . Then we can apply the same method as in lemma 2.7 to obtain

$$\|e^{\delta t} \dot{u}_s\|_{C^0} < C,$$

where  $C$  does not depend on  $s$ . Hence we have an  $C^0$ -estimate of  $e^{\delta t}u_s$  since  $\int_0^s \dot{u}_s ds = u_s$ .

*proof of the existence theorem 1.5* We obtain a  $C^{2,\alpha}$ -estimate of  $u$  by applying the general method of the 2-nd order elliptic differential equations to our conical Kähler manifolds as in the case of compact Kähler manifolds. This method was developed in [8], [17]. (for instance, see [28]). Successively we apply the Schauder estimate to obtain the  $C^{k+2,\alpha}$ -estimate of  $u$  (see Theorem 6.2 and Theorem 17.15 in [13]). This procedure is explained in page 89 [23]. We have the equation,

$$\Delta_s e^{\delta t} \dot{u}_s = H \quad (2.49)$$

where the  $C^{k,\alpha}$ -norm of  $H$  is bounded. The  $C^{k,\alpha}$ -norms of coefficients of the Laplacian  $\Delta_s$  are bounded. We already have the bound of  $C_\delta^0$ -norm of  $u$ . Then applying the Schauder estimate to the equation (2.49), we obtain an estimate of the weighted norms  $C_\delta^{k+2,\alpha}$  of  $u$ . Hence it follows that the set  $S$  is closed by the Ascoli-Arzelà lemma. Since  $S$  is open, it follows that  $S = [0, 1]$ . Thus we have a solution of the Monge-Ampère which gives the Ricci-flat conical Kähler metric on  $X$ . q.e.d.

*proof of the theorem 1.7* Since  $\omega' = \omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u$ , and both  $\omega$  and  $\omega'$  satisfy the Monge-Ampère equation, then we have  $\Omega \wedge \bar{\Omega} = c_n \omega^n = c_n \omega_u^n$ . It follows that  $F \equiv 1$ . Then apply the inequality of the proposition 2.10 for  $\delta p > 2n - 2$ , we have

$$\int_X |(d|u|^{\frac{n}{2}})|_g^2 \omega^n = 0.$$

Hence  $u \equiv \text{constant}$  and it follows  $\omega = \omega'$ . q.e.d.

*proof of theorem 1.8* We set  $\alpha = \omega - \omega'$ . Then the  $d$ -exact 2-form  $\alpha$  decays with the order  $O(e^{-\lambda t})$ , i.e.,  $\alpha \in C_\lambda^k$  with respect to the cylindrical metric. Then there is a  $d$ -closed form  $\alpha_{\text{cpt}}$  with compact support such that  $\alpha - \alpha_{\text{cpt}} = dp$ , with a 1-form  $p \in C_\lambda^k$ . Since  $H^1(S) = \{0\}$ , we have the exact sequence,

$$0 \rightarrow H_{\text{cpt}}^2(X) \rightarrow H^2(X) \rightarrow H^2(S) \rightarrow \dots$$

Since  $\alpha$  is  $d$ -exact, it follows from the exact sequence that  $[\alpha_{\text{cpt}}] = 0 \in H_{\text{cpt}}^2(X)$ . It implies that there is a 1-form  $\theta \in C_\lambda^k$  with  $\alpha = d\theta$ . When we decompose  $\theta = \theta^{1,0} + \theta^{0,1}$ , we have  $\alpha = d\theta^{0,1} + \overline{d\theta^{0,1}}$  and  $\bar{\partial}\theta^{0,1} = 0$ , since  $\alpha$  is a form of type  $(1,1)$ . Let  $\Delta_\omega$  be the Laplacian and  $\square_\omega$  the complex Laplacian with respect to the conical Kähler metric  $\omega$ . Since  $\omega$  is Kählerian, we have  $\Delta_\omega = 2\square_\omega$ . Let  $d_{\text{cone}}^*$  be the adjoint operator of  $d$  and  $\bar{\partial}_{\text{cone}}^*$  the adjoint operator of  $\bar{\partial}$  with respect to the conical metric  $\omega$ . Since  $\lambda > n$ , we can apply the Stokes theorem to obtain

$$\ker \Delta_\omega = \ker \square_\omega \quad (2.50)$$

$$= \ker \bar{\partial} \cap \ker \bar{\partial}_{\text{cone}}^* \quad (2.51)$$

$$= \ker d \cap \ker d_{\text{cone}}^* \quad (2.52)$$

Since the operator  $P = r^2 \square_\omega : C_\lambda^{k+2} \rightarrow C_\lambda^k$  is a cylindrical elliptic operator which becomes Fredholm for generic  $\lambda$  as in before. Since the multiplication by  $r^{-2}$  gives the isomorphism  $C_\lambda^k \rightarrow C_{\lambda+2}^k$ . Hence the Laplacian  $\square_\omega = r^{-2}P : C_\lambda^{k+2} \rightarrow C_{\lambda+2}^k$  is a Fredholm operator for generic  $\lambda$  and we have the Hodge decomposition for the Laplacian  $\square_\omega$ . We denote by  $\text{Im } \bar{\partial}_{\text{cone}}^*$  the image of the operator  $\bar{\partial}_{\text{cone}}^*$ . Since  $\lambda > n$ , we apply the Stokes theorem again to show that  $\text{Im } \bar{\partial}_{\text{cone}}^* \cap \ker \bar{\partial} = \{0\}$ . Then the form  $\theta^{0,1}$  of type  $(0,1)$  is decomposed into

$$\theta^{0,1} = h + \bar{\partial}u,$$

where  $h$  is a harmonic 1-form which is  $d$ -closed, and  $u$  is a function in  $C_\lambda^k$ . Hence  $\alpha = d\theta^{0,1} + \bar{d}\theta^{0,1} = 2\sqrt{-1}\partial\bar{\partial}u^{Im}$ , where  $u^{Im} = (2\sqrt{-1})^{-1}(u - \bar{u})$ . Since  $u^{Im} \in C_\lambda^k$ , for a positive constant  $\lambda$ , the result follows from the theorem 1.7. q.e.d.

### 3 Sasakian structures and Kähler structures on the cone

We will give a brief explanation of Sasakian manifolds from a view point of a correspondence to Kähler structures on the cones. A notion of Sasaki manifolds was introduced by Sasaki and now there is a good reference on Sasaki geometry[4] in which the material in this section can be found. Let  $S$  be a compact manifold of dimension  $2n - 1$ . Note that  $S$  does not have a boundary. The cone of  $S$  is the product  $\mathbb{R}_{>0} \times S$  with  $r = e^t \in \mathbb{R}_{>0}$ . A Riemannian metric  $g_S$  on  $S$  yields the cone metric  $g$  on  $C(S)$  by

$$g = dr^2 + r^2 g_S \tag{3.1}$$

In this section a metric on  $C(S)$  is always the cone metric  $g$  which is given by a metric  $g_S$  on  $S$

**Definition 3.1** *A  $(2n - 1)$  dimensional Riemannian manifold  $(S, g_S)$  is a Sasakian manifold if there is a complex structure  $J$  on the cone such that  $(C(S), g, J)$  is a Kähler manifold which satisfies  $L_{r \frac{\partial}{\partial r}} J = 0$ , where  $r \frac{\partial}{\partial r} = \frac{\partial}{\partial t}$  is the vector field on  $C(S)$  which is defined by the translation of the cone in terms of the cylinder parameter  $t$ . The Lie derivative  $L_{r \frac{\partial}{\partial r}} J = 0$  implies that  $J$  is invariant under the translations with respect to  $t$ , in other words, the vector field  $\frac{\partial}{\partial t}$  is the real part of a holomorphic vector field.*

This is a relevant definition of Sasakian manifolds focusing on the relation to Kähler geometry, however which is different from the ordinary definition.

We shall explain the correspondence to the ordinary one in which interesting geometric structures are included. The Kähler structure  $(C(S), g, J, \omega)$  as in definition 3.1 gives geometric structures on  $S$ . At first we regard  $S$  as the hypersurface  $\{x \in C(S) \mid r(x) = 1\}$  in  $C(S)$ . We define a vector field  $\xi = Jr \frac{\partial}{\partial r} = J \frac{\partial}{\partial t}$  on  $C(S)$ . Since  $g$  is a Hermitian metric,  $\xi$  is orthogonal to  $\frac{\partial}{\partial t}$  which implies that  $\xi$  is a vector field along the hypersurface  $S$ . The restriction of  $\xi$  to  $S$  is denoted by

$$\xi_S = \xi|_S$$

We have the complex structure  $J^*$  which acts on 1-forms by  $(J^*\theta)(v) = \theta(Jv)$ , for  $\theta \in T^*C(S)$  and  $v \in TC(S)$ . Then a 1-form  $\eta$  on  $C(S)$  is given by

$$\eta = -J^* \frac{dr}{r} = -J^* dt$$

and we denote by  $\eta_S$  the restriction of  $\eta$  to  $S$ . (For simplicity, we write  $J$  for  $J^*$  from now on.) Then it follows from the definition that  $\eta(\xi) = \eta_S(\xi_S) = 1$ . The Kähler form  $\omega$  is defined by  $\omega(u, v) = g(Ju, v)$ , for  $u, v \in TC(S)$ . The Lie derivative  $L_{\frac{\partial}{\partial t}}$  is induced from the translation of  $t$ -direction on  $C(S)$ . The group of one parameter transformations  $f_\lambda$  on  $C(S)$  is given by

$$f_\lambda(r, y) = (\lambda r, y), \quad y \in S$$

and

$$\frac{d}{d\lambda} f_\lambda^*|_{\lambda=0} = L_{r \frac{\partial}{\partial r}}$$

From (3.1), we have  $f_\lambda^* g = \lambda^2 g$  and  $L_{r \frac{\partial}{\partial r}} g = 2g$ . Since  $L_{r \frac{\partial}{\partial r}} J = 0$ , we have  $L_{r \frac{\partial}{\partial r}} \omega = 2\omega$ . Since  $d\omega = 0$ , applying the formula of the Lie derivative  $L_u = i_u d + di_u$ , we have

$$di_{r \frac{\partial}{\partial r}} \omega = 2\omega \quad (3.2)$$

Then we have

**Lemma 3.2**

$$i_{r \frac{\partial}{\partial r}} \omega = i_\xi g = r^2 \eta$$

*proof* Since  $\xi = J \frac{\partial}{\partial t}$ , and  $\eta = -J^* dt$ , we have  $\eta(\xi) = 1$ . Then  $i_\xi g(\xi) = r^2 \eta(\xi) = r^2$ . Since  $J \frac{\partial}{\partial t} \in TS$ , we obtain  $\eta(\frac{\partial}{\partial t}) = -dt(J \frac{\partial}{\partial t}) = 0$ . Let  $\langle \frac{\partial}{\partial t}, J \frac{\partial}{\partial t} \rangle^\perp$  be the orthogonal subspace to the space spanned by two vector fields  $\frac{\partial}{\partial t}$  and  $J \frac{\partial}{\partial t}$ . Then  $\langle \frac{\partial}{\partial t}, J \frac{\partial}{\partial t} \rangle^\perp$  is invariant under the action of  $J$  and we have  $\eta(u) = -dt(Ju) = 0$  for  $u \in \langle \frac{\partial}{\partial t}, J \frac{\partial}{\partial t} \rangle^\perp$ . Thus  $i_\xi g = r^2 \eta$  q.e.d.

Applying lemma 3.2 to the equation (3.2), since  $\eta = -J^* \frac{dr}{r}$ , we have

$$2\omega = dr^2 \eta = -d(J^* r dr) = -\frac{1}{2} dJ^* dr^2 \quad (3.3)$$

Hence

$$\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2$$

Since  $2\omega = 2rdr \wedge \eta + r^2 d\eta$  is a symplectic structure on  $C(S)$ ,

$$(2\omega)^n = 2nr^{2n} dt \wedge \eta \wedge (d\eta)^{n-1} \neq 0$$

It implies that  $\eta_S \wedge (d\eta_S)^{n-1} \neq 0$ . We see that  $\eta_S$  is a contact structure on  $S$  and  $\xi_S$  is the Reeb vector field. We define a distribution  $D$  of dimension  $2n-2$  by  $D = \ker \eta_S = \{u \in TS \mid \eta_S(u) = 0\}$ , then  $D = \langle \frac{\partial}{\partial t}, J \frac{\partial}{\partial t} \rangle^\perp$  and  $D$  is invariant under the action of  $J$ . We define a section  $\Phi_S \in \text{End}(TS)$  by

$$\Phi_S(v) = \begin{cases} Jv & (v \in D) \\ 0 & (v = \xi) \end{cases}$$

Together with the contact structure  $\eta_S$  and the Reeb vector field  $\xi_S$  on  $S$ ,  $\Phi_S \in \text{End}(TS)$  gives the Riemannian metric on  $S$  by

$$g_S(u, v) = \eta_S \otimes \eta_S(u, v) + d\eta_S(u, \Phi_S v), \quad (3.4)$$

for  $u, v \in TS$ . By the same procedure, an almost Kähler structure on  $C(S)$  as in definition 3.1 gives the structure  $(\eta_S, \xi_S, \Phi_S)$  on  $S$ . When  $J$  is integrable, the corresponding structure on  $S$  admits suitable properties.

**Lemma 3.3** *If a almost complex structure  $J$  on  $C(S)$  is integrable, we have*

$$L_\xi J = J L_{\frac{\partial}{\partial t}} J$$

*proof* Since  $J$  is integrable, the Nijenhuis tensor vanishes,

$$[J \frac{\partial}{\partial t}, Ju] = J[J \frac{\partial}{\partial t}, u] + J[\frac{\partial}{\partial t}, Ju] + [\frac{\partial}{\partial t}, u] \quad (3.5)$$

where  $u \in TC(S)$ . Since  $\xi = J \frac{\partial}{\partial t}$ , we have

$$L_\xi J(u) = [J \frac{\partial}{\partial t}, Ju] - J[J \frac{\partial}{\partial t}, u] \quad (3.6)$$

$$L_{\frac{\partial}{\partial t}} J(u) = [\frac{\partial}{\partial t}, Ju] - J[\frac{\partial}{\partial t}, u] \quad (3.7)$$

Thus from (3.5), we have  $L_\xi J = J L_{\frac{\partial}{\partial t}} J$ . q.e.d.

Then  $\frac{\partial}{\partial t} - \sqrt{-1} J \frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \sqrt{-1} \xi$  is a holomorphic vector field on  $C(S)$ . In particular,  $[\frac{\partial}{\partial t}, \xi] = [\frac{\partial}{\partial t}, J \frac{\partial}{\partial t}] = 0$ .

**Lemma 3.4** *If  $L_\xi J = 0$ , then  $L_{\xi_S} \eta_S = 0$ .*

*proof* Since  $L_\xi \eta = -L_\xi J dt = -J L_{\frac{\partial}{\partial t}} dt = 0$ , the result follows from  $i_S^* L_\xi \eta = L_{\xi_S} \eta_S$ . q.e.d.

Thus if  $J$  is integrable, then  $L_\xi \eta = 0$ . This also implies that  $i_\xi d\eta = 0$  and  $d\eta_S$  is a basic form on  $S$ . Since  $L_{\xi_S} \Phi_S = 0$ , (3.4) yields  $L_{\xi_S} g_S = 0$  and we see that  $\xi_S$  is a Killing vector field on  $(S, g_S)$ .

Hence we obtain the structure  $(\eta_S, \xi_S, \Phi_S)$  on  $S$  from a Kähler structure with  $L_{\frac{\partial}{\partial t}} J = 0$  and a cone metric. Conversely we shall construct the Kähler structure as in definition 3.1 on  $C(S)$  from the structure  $(\eta_S, \xi_S, \Phi_S)$  on  $S$ . Let  $\eta_S$  be a contact structure on  $S$ . Since the Darboux's theorem gives the standard form of a contact structure, it follows that  $D = \ker \eta_S$  is a  $2n - 2$  dimensional distribution on which  $d\eta_S$  is a non-degenerate 2-form.

Then a section  $\Phi_S \in \text{End}(TS)$  is defined as an almost complex structure on  $D$  with  $\Phi_S(\xi_S) = 0$ . Such a section  $\Phi_S$  is compatible with  $\eta_S$  if the following condition hold

- $d\eta_S(\Phi_S u, \Phi_S v) = d\eta_S(u, v), \quad u, v \in D$
- $d\eta_S(u, \Phi_S u) > 0, \quad (u \neq 0 \in D).$

These conditions imply that a pair  $(d\eta_S, \Phi_S)$  is a Hermitian structure on  $D$ . Thus a compatible pair  $(\eta_S, \Phi_S)$  gives a Riemannian metric  $g_S$  by (3.4) on  $S$ . Then we have the cone metric  $g$  on the cone  $C(S) = \mathbb{R}_{>0} \times S$  by

$$g = dr^2 + r^2 g_S$$

The tangent bundle  $TC(S)$  is decomposed into  $T\mathbb{R} \times TS$  and we regarded  $\xi_S \in TS$  as the vector field  $\xi$  on  $C(S)$ . An almost complex structure  $J$  on  $C(S)$  is given by

$$J(r \frac{\partial}{\partial r}) = \xi, \quad J|_D = \Phi_S$$

Then  $g$  is a Hermitian metric with respect to  $J$  and the corresponding 2-form  $\omega$  is a symplectic form,

$$2\omega = d(r^2 \eta_S) = 2r dr \wedge \eta_S + r^2 d\eta_S$$

that is,  $(g, J, \omega)$  is an almost Kähler structure on  $C(S)$ . Since  $\Phi_S$  is a section of  $\text{End}(TS)$ , the induced  $J$  is invariant under the translation with respect to  $t$ . Thus  $L_{\frac{\partial}{\partial t}} J = 0$ . Hence we obtain the almost Kähler structure as in definition 3.1 from the structure  $(\eta_S, \xi_S, \Phi_S)$ .

**Definition 3.5** *Let  $S$  be a manifold of dimension  $2n - 1$  which admits a contact structure  $\eta_S$  and a compatible structure  $\Phi_S$  to  $\eta_S$ . A compatible pair  $(\eta_S, \Phi_S)$  is a Sasakian structure on  $S$  if the corresponding almost Kähler structure  $(C(S), g, J, \omega)$  is Kählerian, that is,  $J$  is integrable.*

Hence our argument is reduced to the following,

**Proposition 3.6** *There is a one to one correspondence between Sasakian structures  $(\eta_S, \xi_S, \Phi_S)$  on  $S$  and Kähler structures  $(g, J, \omega)$  on  $C(S)$  consisting of a cone metric  $g$  and a translation-invariant complex structure  $J$ .*

## 4 Einstein-Sasakian structures and weighted Calabi-Yau structures

As in section 1, let  $S$  be a compact manifold of dimension  $2n - 1$  and  $C(S)$  the cone  $\mathbb{R}_{>0} \times S$  with  $r = e^t \in \mathbb{R}_{>0}$ . We assume that  $S$  is simply connected in this section. Let  $(C(S), J, g, \omega)$  be a Kähler structure on  $C(S)$  corresponding to a Sasakian structure on  $S$  by the proposition 3.6. Then as in section 1, we have

$$\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2 \quad (4.1)$$

$$g = dr^2 + r^2 g_S \quad (4.2)$$

$$\mathbb{L}_{\frac{\partial}{\partial t}} J = 0 \quad (4.3)$$

We also see in section 1

$$2\omega = d(r^2 \eta_S) = 2r dr \wedge \eta_S + r^2 d\eta_S \quad (4.4)$$

We assume that the canonical line bundle of  $(C(S), J)$  is trivial and there is a nowhere-vanishing holomorphic  $n$ -form  $\Omega$ .

We call an  $n$ -form  $\Omega$  is of weight  $n$  if  $\Omega$  satisfies

$$\mathbb{L}_{\frac{\partial}{\partial t}} \Omega = n\Omega \quad (4.5)$$

Note that  $\omega$  and  $g$  satisfy

$$\mathbb{L}_{\frac{\partial}{\partial t}} \omega = 2\omega \quad (4.6)$$

$$\mathbb{L}_{\frac{\partial}{\partial t}} g = 2g \quad (4.7)$$

and in this sense these are of weight 2. Note that a pair  $(\Omega, \omega)$  consisting of a holomorphic  $n$ -form and a Kähler form induces a Kähler structure  $(J, \omega)$  on  $C(S)$ .

**Definition 4.1** A pair consisting of a  $d$ -closed, holomorphic  $n$ -form  $\Omega$  of weight  $n$  and a Kähler form  $\omega$  on  $C(S)$  is a weighted Calabi-Yau structure if the induced Kähler structure  $(J, \omega)$  corresponds to a Sasakian structure on  $S$  as in proposition 3.6 and satisfies the Monge-Ampère equation,

$$\Omega \wedge \sigma(\bar{\Omega}) = \frac{(2\sqrt{-1})^n}{n!} \omega^n, \quad (4.8)$$

where  $\bar{\Omega}$  is the complex conjugate of  $\Omega$  and  $\sigma(\bar{\Omega})$  is given by

$$\sigma(\bar{\Omega}) = \begin{cases} +\bar{\Omega}, & n \equiv 0, 1 \pmod{4}, \\ -\bar{\Omega}, & n \equiv 2, 3 \pmod{4}. \end{cases}$$

We regard  $S$  as the hypersurface  $\{t = 0\} = \{r = 1\}$  as before.

Let  $i_S : S \rightarrow C(S)$  be the inclusion of  $S$  into  $C(S)$  and  $p_S : C(S) \rightarrow S$  the projection to  $S$ . For a weighted Calabi-Yau structure  $(\Omega, \omega)$ , using the interior product by the vector field  $\frac{\partial}{\partial t}$ , we define

$$\psi = i_{\frac{\partial}{\partial t}} \Omega, \quad (4.9)$$

$$r^2 \eta = i_{\frac{\partial}{\partial t}} \omega, \quad (4.10)$$

and restricting to  $S$ , we have  $\psi_S := i_S^* \psi = i_S^* i_{\frac{\partial}{\partial t}} \Omega = i_S^* \eta$ . Since the vector field  $\frac{\partial}{\partial t}$  is the real part of a holomorphic vector field, it follows that  $\psi$  is a holomorphic  $(n-1)$ -form on  $C(S)$ , however  $\psi$  is not  $d$ -closed. Since  $\Omega$  is a  $d$ -closed form of weight  $n$ , we have

$$L_{\frac{\partial}{\partial t}} \Omega = di_{\frac{\partial}{\partial t}} \Omega = d\psi = n\Omega \quad (4.11)$$

Since  $(dt - \sqrt{-1}\eta) = dt + \sqrt{-1}Jdt$  is a form of type  $(1, 0)$  it follows from (4.9) that

$$\Omega = (dt - \sqrt{-1}\eta) \wedge \psi \quad (4.12)$$

Restricting to  $S = \{t = 0\}$ , we have

$$d\psi_S = -n\sqrt{-1}\eta_S \wedge \psi_S$$

Since  $(\Omega, \omega)$  satisfies the Monge-Ampère equation, substituting (4.12), we have

$$\begin{aligned} \Omega \wedge \sigma(\bar{\Omega}) &= (dt - \sqrt{-1}\eta) \wedge \psi \wedge \sigma(\bar{\psi}) \wedge (dt + \sqrt{-1}\eta) \\ &= 2\sqrt{-1}dt \wedge \eta \wedge \psi \wedge \sigma(\bar{\psi}) \end{aligned}$$

it also follows from (4.4)

$$\frac{(2\sqrt{-1})^n}{n!} \omega^n = \frac{(\sqrt{-1})^n}{n!} 2nr^{2n} dt \wedge \eta \wedge (d\eta)^{n-1}$$

Then by applying the interior product  $i_{\frac{\partial}{\partial t}}$  to both sides of the Monge-Ampère equation, we have

$$\eta \wedge \psi \wedge \sigma(\overline{\psi}) = c_{n-1} r^{2n} \eta \wedge \left(\frac{1}{2} d\eta\right)^{n-1} \quad (4.13)$$

By restricting to  $S = \{r = 1\}$ , we have

$$\eta_S \wedge \psi_S \wedge \sigma(\overline{\psi}_S) = c_{n-1} \eta_S \wedge \left(\frac{1}{2} d\eta_S\right)^{n-1} \quad (4.14)$$

Let  $D$  be the subbundle of  $TS$  given by the kernel of  $d\eta_S$ . Then  $\psi_S$  defines a section  $\Phi_S$  on  $\text{End}(D)$  which is an almost complex structure and then  $\psi_S$  is a nowhere vanishing section of the canonical line bundle  $K_D$  of  $D$ . Since  $\omega$  is Kählerian, we see that  $d\eta_S$  is a Hermitian form on  $D$  and then  $(\eta_S, \Phi_S)$  is an Sasakian structure on  $S$ . An Einstein-Sasakian structure on  $S$  is a Sasakian structure whose metric  $g$  is an Einstein metric on  $S$ .

**Proposition 4.2 .** *Let  $(\Omega, \omega)$  be a weighted Calabi-Yau structure on  $C(S)$ . Then  $(\Omega, \omega)$  corresponds to an Einstein-Sasakian structure on  $S$  under the correspondence in section 1. Conversely, an Einstein-Sasakian structure on  $S$  gives a weighted Calabi-Yau structure  $(\Omega, \omega)$  on  $C(S)$ , that is, there is a one to one correspondence between Einstein-Sasakian structures on  $S$  and weighted Calabi-Yau structures on the cone  $C(S)$ .*

As though a correspondence as in the proposition is already known among experts in Sasakian geometry (see [4] for instance), our point of view may be rather relevant since we focus on weighted Calabi-Yau structures on  $C(S)$  which exactly correspond to Einstein-Sasakian structures on  $S$ . In the appendix we shall give a proof.

## 5 Ricci-flat conical Kähler metrics on crepant resolutions of normal isolated singularities

Let  $X_0$  be an affine variety of complex dimension  $n$  with only normal isolated singularity  $p \in X_0$ . In this section, we assume that the complement  $X_0 \setminus \{p\}$  is biholomorphic to the cone  $C(S)$  of an Einstein-Sasakian manifold  $S$  of real dimension  $2n - 1$ . As we see in section 3 and 4, there is the weighted Calabi-Yau structure  $(\Omega, \omega_0)$  on the cone  $C(S)$  and the Ricci-flat Kähler cone metric  $\omega_0$  which is given by

$$\omega_0 = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2 \quad (5.1)$$

**Theorem 5.1** *Let  $X_0$  be an affine variety with only normal isolated singularity  $\{p\}$ . We assume that the complement  $X_0 \setminus \{p\}$  is biholomorphic to the cone  $C(S)$  of an Einstein-Sasakian manifold  $S$  of real dimension  $2n - 1$ . If there is a resolution of singularity  $\pi : X \rightarrow X_0$  with trivial canonical line bundle  $K_X$ , there is a Ricci-flat conical Kähler metric for every Kähler class of  $X$ .*

**Remark 5.2** *Van Coevering [7] showed that there is a Ricci-flat conical Kähler metric in the Kähler class which belongs to the compactly supported cohomology group  $H_{\text{cpt}}^2(X)$  of  $X$ . Our theorem shows that this kind of restricted condition is not necessary and implies that the conjecture on the existence of complete Ricci-flat Kähler metrics in [20] is affirmative.*

The Ricci-flat Kähler cone metric  $\omega_0$  is written as

$$\omega_0 = e^{2t}(2dt \wedge d^c t + dd^c t), \quad (5.2)$$

where  $dd^c t$  is the transversal Kähler metric on the Sasakian manifold  $S$  and the Reeb vector field  $\xi_S$  is given by  $J \frac{\partial}{\partial t}$  which gives the Reeb foliation on  $S$ . Let  $H_B^p(S)$  be the basic cohomology group on  $S$  with respect to the Reeb foliation. There are the Hodge decomposition of the basic cohomology groups and the Lefschetz decomposition as in Kähler geometry. Thus the second basic cohomology group  $H_B^2(S)$  is decomposed into the followings,

$$H_B^2(S, \mathbb{C}) = H_B^{1,1}(S) \oplus H_B^{2,0}(S) \oplus H_B^{0,2}(S), \quad (5.3)$$

$$H_B^{1,1}(S, \mathbb{R}) = \mathbb{R}dd^c t \oplus P_B^{1,1}(S, \mathbb{R}), \quad (5.4)$$

where  $H_B^{p,q}(S)$  denote the basic Dolbeault cohomology group of type  $(p, q)$  and  $\mathbb{R}dd^c t$  is the one dimensional space generated by the transversal Kähler form  $dd^c t$  and  $P_B^{1,1}(S)$  is the basic primitive cohomology group of type  $(1, 1)$  with respect to  $dd^c t$ . Since the transversal complex structure on  $S$  admits the positive first Chern class which implies that the anti-canonical line bundle on  $S$  is positive. Then by the same procedure as in Kähler manifolds, we obtain the vanishing theorem of Kodaira and the Serre duality of the basic cohomology groups. Then we have the vanishing of cohomology groups,

**Lemma 5.3**

$$H_B^{2,0}(S) = H_B^{0,2}(S) = \{0\}.$$

*proof* Since  $S$  is a real  $2n - 1$  dimensional Sasakian manifold,  $S$  admits the transversal complex structure of dimension  $n - 1$ . Applying the Serre duality and the Kodaira vanishing theorem for the negative  $K_S$ , we have

$$\begin{aligned} H_B^{0,2}(S) &= H_B^2(S, \mathcal{O}_B) = H^{n-3}(S, K_S) = \{0\}, \\ H_B^{0,2}(S) &= \overline{H_B^{2,0}(S)} = \{0\} \end{aligned}$$

q.e.d.

**Lemma 5.4**

$$H_B^2(S) = H_B^{1,1}(S) = \mathbb{R}dd^c t \oplus H^2(S, \mathbb{R})$$

*proof* It is shown that the de Rham cohomology group  $H^i(S)$  of the Sasakian manifold  $S$  coincides with the basic primitive cohomology groups  $P_B^i(S)$ , for  $1 \leq i \leq n-1$  (for instance, see the proposition 7.4.13, pp. 233 in [4]). It follows from the lemma 5.3 that

$$H_B^2(S) = H_B^{1,1}(S) = \mathbb{R}dd^c t \oplus P_B^2(S) = \mathbb{R}dd^c t \oplus H^2(S)$$

q.e.d.

**Lemma 5.5** *Let  $\pi : X \rightarrow X_0$  be a resolution of an affine variety  $X_0$  with only normal isolated singularity with trivial  $K_X$ . Then we have the vanishing*

$$H^i(X, \mathcal{O}_X) = \{0\}, \quad (\forall i > 0). \quad (5.5)$$

*Further let  $C(S)$  be the complement  $X \setminus E$  where  $E$  is the exceptional set of the resolution  $X$ . Then we also have*

$$H^i(C(S), \mathcal{O}_{C(S)}) = \{0\}, \quad (n-1 > \forall i > 0). \quad (5.6)$$

*In particular, if  $n \geq 3$ , we have*

$$H^1(C(S), \mathcal{O}_{C(S)}) = \{0\},$$

*where  $\dim_{\mathbb{C}} X = n$ .*

*proof* Since  $X_0$  is an affine variety and  $H^p(X_0, R^q \pi_* \mathcal{O}_X) = \{0\}$  for  $p > 0$ , the first vanishing (5.5) follows from the Grauert-Riemenschneider vanishing theorem,

$$R^i \pi_* K_X = R^i \pi_* \mathcal{O}_X = 0, \quad (i > 0)$$

Let  $H_E^i(X, \mathcal{O}_X)$  be the local cohomology groups with supports in  $E$ , and coefficients in the structure sheaf  $\mathcal{O}_X$ . Then the short exact sequence,

$$0 \rightarrow \Gamma_E(X, \mathcal{O}_X) \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(C(S), \mathcal{O}_{C(S)}) \rightarrow 0$$

yields the long exact sequence,

$$\cdots \rightarrow H_E^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^i(C(S), \mathcal{O}_{C(S)}) \rightarrow H_E^{i+1}(X, \mathcal{O}_X) \rightarrow \cdots \quad (5.7)$$

By applying the local duality theorem of the cohomology groups to the resolution  $\pi : X \rightarrow X_0$ , we have

$$\dim H_E^j(X, \mathcal{O}_X) = \dim R^{n-j}\pi_* K_X = \dim R^{n-j}\pi_* \mathcal{O}_X.$$

Then it follows from the Grauert-Riemenschneider vanishing theorem that

$$H_E^j(X, \mathcal{O}_X) = \{0\}$$

for all  $n - j > 0$ . Thus from the exact sequence (5.7) and (5.5), we obtain

$$H^i(C(S), \mathcal{O}_{C(S)}) = \{0\}, \quad (n - 1 > \forall i > 0).$$

It implies that  $H^1(C(S), \mathcal{O}_{C(S)}) = \{0\}$  if  $n \geq 3$ . q.e.d.

The resolution  $\pi : X \rightarrow X_0$  gives the identification  $\pi|_{X \setminus E} : X \setminus E \cong X_0 \setminus \{p\} = C(S)$ . We denote by  $\{t > T\}$  the subset of the cone  $C(S) = \mathbb{R} \times S$  defined by

$$\{t > T\} := \{(t, s) \in \mathbb{R} \times S \mid t > T\},$$

for a constant  $T$ . The subset  $\{t > T\}$  is regarded as the subset of  $X$  by the identification  $\pi|_{X \setminus E} : X \setminus E \cong X_0 \setminus \{p\} = C(S)$ . Then the set  $\{t > T\}$  is an unbounded region of  $X$  and we also denote by  $\{t \leq T\}$  the complement of  $\{t > T\}$  in  $X$  which is a compact set including the exceptional set  $E$ . Let  $p_S$  be the projection from  $C(S) = \mathbb{R} \times S$  to  $S$ . We use the same notation  $p_S$  for the restricted projection  $\{t > T\} \rightarrow S$ .

**Lemma 5.6** *Let  $\kappa$  be an arbitrary Kähler form on  $X$  with the Kähler class  $[\kappa] \in H^2(X, \mathbb{R})$ . We assume  $\dim_{\mathbb{C}} X \geq 3$ . Then there is a real form  $\tilde{\kappa}$  of type  $(1, 1)$  on  $X$  which satisfies the followings,*

- (i)  $[\tilde{\kappa}] = [\kappa] \in H^2(X, \mathbb{R})$
- (ii) *For a sufficiently large constant  $T_+$ , the restriction  $\tilde{\kappa}$  to the subset  $\{t > T_+\}$  is given by the pull back of a  $d$ -closed, primitive basic form  $\hat{\kappa}$  of type  $(1, 1)$  on  $S$ , i.e.,*

$$\tilde{\kappa}|_{\{t > T\}} = p_S^* \hat{\kappa}.$$

*proof* It follows from the lemma 5.3 and 5.4 that we have

$$H^2(C(S)) \cong H^2(S) \cong P_B^2(S) \cong P_B^{1,1}(S)$$

Then there is a  $d$ -closed, primitive basic form  $\hat{\kappa} \in P_B^{1,1}(S)$  and a 1-form  $\theta$  on  $C(S)$  such that

$$\kappa|_{C(S)} = p_S^* \hat{\kappa} - d\theta. \tag{5.8}$$

Then  $d\theta$  is a form of type  $(1, 1)$  on  $C(S)$  where  $\theta = \theta^{1,0} + \theta^{0,1}$  and we have

$$d\theta = \bar{\partial}\theta^{1,0} + \partial\theta^{0,1}, \quad \partial\theta^{1,0} = 0, \quad \bar{\partial}\theta^{0,1} = 0$$

It follows from the lemma 5.5 that there is a function  $\phi$  on  $C(S)$  such that  $\theta^{0,1} = \bar{\partial}\phi$ . Since  $\theta$  is a real form, it follows from  $\theta^{1,0} = \overline{\theta^{0,1}}$  that

$$\begin{aligned} d\theta &= \partial\theta^{0,1} + \overline{\partial\theta^{0,1}} \\ &= \partial\bar{\partial}\phi + \bar{\partial}\partial\bar{\phi} \\ &= 2\sqrt{-1}\partial\bar{\partial}\phi^{Im} \end{aligned}$$

where  $\phi^{Im} = \frac{1}{2\sqrt{-1}}(\phi - \bar{\phi})$  is the imaginary part of  $\phi$ . Let  $\rho_T$  be a smooth function such that

$$\rho_T(x) = \begin{cases} 1 & x \in \{T < t\} \\ 0 & x \in \{t \leq T - 1\} \end{cases}$$

and  $0 \leq \phi_T(x) \leq 1$  on  $X$ , i.e.,  $\rho_T$  is a cut off function which takes the value 0 on a neighborhood of  $E$  and 1 at the infinity. We define  $\tilde{\kappa}$  by

$$\tilde{\kappa} = k + 2\sqrt{-1}\partial\bar{\partial}\rho_{T_+}\phi^{Im}.$$

Then we see that  $\tilde{\kappa}$  is a form of type  $(1, 1)$  on  $X$  which is the Kähler form  $\kappa$  on  $\{t \leq T_+ - 1\}$  and  $p_s^*\hat{\kappa}$  on  $\{t > T_+\}$  from (5.8). Thus  $\tilde{\kappa}$  satisfies the conditions. q.e.d.

**Lemma 5.7** *Let  $\kappa$  be an arbitrary Kähler form on  $X$ . We assume  $\dim_{\mathbb{C}} X \geq 3$ . Then there exists a Kähler form  $\omega_{\kappa,0}$  which satisfies the followings,*

- (i)  $[\omega_{\kappa,0}] = [\kappa] \in H^2(X, \mathbb{R})$ .
- (ii) *There are constants  $T_+$  and  $T_-$  such that  $\omega_{\kappa,0}$  coincides with  $c\omega_0 + \tilde{\kappa}$  on the region  $\{1 + T_- < t\}$ , where  $c$  is a positive constant and  $\omega_0$  is the Kähler cone metric on  $C(S)$  as before and  $\tilde{\kappa}$  is the form in lemma 5.6, where  $1 + T_- < T_+$*

*proof* We see that there is a positive function  $\psi(x)$  on  $\mathbb{R}$  which satisfies

$$\psi(x) = \begin{cases} x & \text{if } x > e^{2(1+T_-)} \\ \text{constant} & \text{if } e^{2T_-} > x \end{cases}$$

and  $\psi'(x) \geq 0$  and  $\psi''(x) \geq 0$  on the region  $\{e^{2T_-} \leq x \leq e^{2(1+T_-)}\}$ . Then the composite function  $\psi(r^2)$  is a function on  $X$  since  $\psi$  is a constant on the region  $\{t < T_-\}$ , where  $r = e^t$  and we have

$$\sqrt{-1}\partial\bar{\partial}\psi(r^2) = \sqrt{-1}\psi''(r^2)\partial r^2 \wedge \bar{\partial} r^2 + \sqrt{-1}\psi''(r^2)\partial\bar{\partial} r^2$$

It implies that  $\sqrt{-1}\partial\bar{\partial}\psi(r^2)$  is semi-positive on the region  $\{t \leq 1+T_-\}$  which is the Kähler cone metric  $\omega_0$  on the region  $\{1+T_- < t\}$ . Thus we define  $\omega_{\kappa,0}$  by

$$\omega_{\kappa,0} = c\sqrt{-1}\partial\bar{\partial}\psi(r^2) + \tilde{\kappa}.$$

Then  $\omega_{\kappa,0}$  satisfies both conditions (i) and (ii). So it suffices to show that  $\omega_{\kappa,0}$  is a Kähler form for constants  $T_+$ ,  $T_-$  and  $c$ , which are taken by the following. We divide  $X$  into the following five regions:

$$(1) \{t < T_-\} \quad (2) \{T_- \leq t \leq 1+T_-\} \quad (3) \{1+T_- < t < T_+ - 1\}, \\ (4) \{T_+ - 1 \leq t \leq T_+\} \quad (5) \{T_+ < t\}$$

On the region (1),  $\omega_{\kappa,0}$  is the Kähler form  $\kappa$  and on the region (3),  $\omega_{\kappa,0}$  is  $c\omega_0 + k$  which is also Kählerian. On the region (2),  $\omega_{\kappa,0} = k + \sqrt{-1}\partial\bar{\partial}\psi(r^2)$  is positive since  $\sqrt{-1}\partial\bar{\partial}\psi(r^2)$  is semi-positive and  $\kappa$  is positive. On the region (5),  $\omega_{\kappa,0}$  is a sum of the Kähler cone form  $\omega_0$  and the bounded form  $p_s^*\hat{k}$  with respect to the cylinder metric. Since the cone metric  $\omega_0$  grows with order  $O(e^{2t})$ , there is a sufficiently large  $T_+ = T_+(c_0)$  for a positive  $c_0$  such that  $c_0\sqrt{-1}\partial\bar{\partial}\psi(r^2) + \tilde{\kappa}$  becomes positive. Finally since the region (4) is compact, there is a positive  $c$  with  $c > c_0$  such that  $\omega_{\kappa,0}$  is a Kähler form on the region (4). We can see that  $\omega_{\kappa,0}$  is still positive for  $c > c_0$  on (5). Hence  $\omega_{\kappa,0}$  is a Kähler form.

q.e.d.

*proof of theorem 5.1* In the case  $\dim_{\mathbb{C}} X = 2$ ,  $X$  is a minimal resolution of the ordinary double point (see [16]). Thus every Kähler class is represented by the class of the exceptional divisors which lies in compactly supported cohomology groups and we can have a Kähler form  $\omega_{\kappa,0}$  as in the lemma 5.7 with  $\hat{k} = 0$ . Hence we have the initial Kähler form  $\omega_{\kappa,0}$  as in lemma 5.7 for every Kähler class on  $X$  of dimension  $n \geq 2$ . We recall that a positive function  $F$  is defined by

$$\Omega \wedge \bar{\Omega} = c_n F_k \omega_{k,0}^n$$

with  $F_k = e^{f_k}$ . We shall show that the function  $F_k$  satisfies

$$\|e^{(2+\delta)t}(F_k - 1)\|_{C^{k,\alpha}} < \infty, \quad \delta > 0, \quad k \geq 3, \quad 0 < \alpha < 1$$

with respect to the cylindrical metric  $dt \wedge d^c t + dd^c t = e^{-2t}\omega_0$ . Then the result follows from theorem 1.5

Since  $\omega_0$  is a Ricci-flat Kähler cone metric, we have

$$\Omega \wedge \bar{\Omega} = c_n \omega_0^n$$

Then  $\omega_0^n = e^{f_k} \omega_{k,0}^n$ . Since  $\omega_{k,0} = \omega_0 + \tilde{\kappa} = \omega_0 + p_S^* \hat{\kappa}$  on the region  $\{t > T_+\}$ , we have

$$e^{-f_k} = \frac{\omega_{k,0}^n}{\omega_0^n} = \frac{(\omega_0 + \tilde{k})^n}{\omega_0^n} \quad (5.9)$$

$$= 1 + n \frac{\tilde{k} \wedge \omega_0^{n-1}}{\omega_0^n} \quad (5.10)$$

$$+ \sum_{i=2}^n \frac{n!}{i!(n-i)!} \frac{\tilde{k}^i \wedge \omega_0^{n-i}}{\omega_0^n} \quad (5.11)$$

Since  $\tilde{k} = p_S^* \hat{\kappa}$  is the pullback of a primitive, basic  $(1,1)$ -form on  $S$  with respect to  $dd^c t$ , we have  $\tilde{k} \wedge (dd^c t)^{n-2} = 0$ . Thus the second term of (5.10) vanishes since we have

$$(n-1)^{-1} \tilde{k} \wedge \omega_0^{n-1} = \tilde{k} \wedge e^{2(n-1)t} (2dt \wedge d^c t \wedge (dd^c t)^{n-2}) \quad (5.12)$$

$$= e^{2(n-1)t} (2dt \wedge d^c t \wedge \hat{\kappa} \wedge (dd^c t)^{n-2}) = 0 \quad (5.13)$$

Each term of (5.11) is given by

$$(n-i)^{-1} \frac{\tilde{k}^i \wedge \omega_0^{n-i}}{\omega_0^n} = \frac{\tilde{k}^i \wedge e^{2(n-i)t} (2dt \wedge d^c t \wedge (dd^c t)^{n-i-1})}{e^{2nt} (2dt \wedge d^c t \wedge (dd^c t)^n)} \quad (5.14)$$

$$= e^{-2it} \frac{\tilde{k}^i \wedge (2dt \wedge d^c t \wedge (dd^c t)^{n-i-1})}{(2dt \wedge d^c t \wedge (dd^c t)^n)} \quad (5.15)$$

Since the 1-form  $dt$  and  $d^c t$  and the 2-form  $dd^c t$ ,  $\tilde{k}$  are bounded with respect to the cylinder metric, we have

$$\frac{\tilde{k}^i \wedge \omega_0^{n-i}}{\omega_0^n} = O(e^{-2it}) = O(r^{-2i})$$

Note that  $\tilde{\kappa} = p_S^* \hat{\kappa}$  for a basic 2-form  $\hat{\kappa}$ . It follows from  $i \geq 2$  that the every term in (5.11) decays with the order  $O(e^{-4t})$ . Hence we have  $e^{-f_k} - 1 = O(e^{-4t})$ . Since  $e^{-f_k}$  is bounded, we obtain

$$e^{f_k} - 1 = O(e^{-4t}).$$

We also have the estimate of the higher order derivative of  $F_k$  since we use the  $C^k$ -norm with respect to the cylinder metric. Note that the  $C^k$ -norm of  $e^{-4t}$  is just estimated the derivative by the cylinder parameter  $t$  which also decays with the order  $O(e^{-4t})$ .

Hence there is a solution  $u$  of the Monge-Ampère equation

$$\Omega \wedge \overline{\Omega} = c_n(\omega_{k,0} + dd^c u)^n$$

from the existence theorem 1.5. Then the Kähler form  $\omega_k = \omega_{k,0} + dd^c u$  gives a Ricci-flat Kähler metric on  $X$  q.e.d.

**Remark 5.8** *If  $\omega_{\kappa,0}$  is not in the form  $\omega_0 + p_S^* \hat{\kappa}$  for the pullback of the basic, primitive  $(1,1)$  form  $\hat{\kappa}$  in the region  $\{t > T_+\}$ ,  $F - 1$  only decays with the order  $O(e^{-2t})$  which we can not apply the existence theorem 1.5*

## 6 Examples of Calabi-Yau structures on crepant resolutions

We construct examples of Ricci-flat conical Kähler metrics. Some of them are already known, however there are new Ricci-flat conical Kähler metrics included whose Kähler classes do not belong to the compactly supported cohomology groups. We start with the trivial example,

**Example 6.1** *The complement  $\mathbb{C}^n \setminus \{0\}$  is the cone of the sphere of dimension  $2n - 1$  and the standard Kähler metric  $\omega_{st}$  on  $\mathbb{C}^n$  is a conical Ricci-flat Kähler metric with  $r^2 = \sum_{i=1}^n |z_i|^2$ . The induced metric on the sphere is an Einstein-Sasakian metric.*

**Example 6.2** *Let  $\Gamma$  be a finite subgroup of the special unitary group  $SU(n)$  which freely acts on  $\mathbb{C}^n \setminus \{0\}$ . Then the quotient  $X_0 = \mathbb{C}^n / \Gamma$  has a normal isolated singularity at the origin 0 and the complement  $\mathbb{C}^n \setminus \{0\} / \Gamma$  is the cone  $C(S)$  of the Einstein-Sasakian manifold  $S := S^{2n-1} / \Gamma$ . The induced metric on the cone  $C(S)$  is a Ricci-flat Kähler cone metric. Then the theorem 5.1 shows that a crepant resolution of the isolated quotient singularity  $\mathbb{C}^n / \Gamma$  has a Ricci-flat conical Kähler metric. This class is already obtained by Kronheimer [16] for  $n = 2$  and by Joyce [15] for  $n > 2$ .*

**Example 6.3** *Let  $Z$  be a compact Kähler-Einstein manifold with positive 1-st Chern class  $c_1(Z)$ , which is called a Fano manifold (We assume that  $\dim_{\mathbb{C}} Z = n - 1 \geq 2$ ). Then the sphere bundle  $S$  of the canonical line bundle  $K_Z$  of the Fano manifold  $Z$  admits an Einstein-Sasakian metric and the complement of the zero-section  $K_Z \setminus \{0\}$  is the cone  $C(S)$  which has the Ricci-flat Kähler cone metric. Then we apply the theorem 5.1 to  $X = K_Z$  and obtain a Ricci-flat conical Kähler metric in every kähler class of  $K_Z$ . Calabi [5]*

already constructed a Ricci flat Kähler metric on  $K_Z$  by the bundle construction whose Kähler class lies in the compactly supported cohomology group, i.e., the anti-canonical class. Since we have the vanishing of the cohomology groups  $H^1(S) = \{0\}$  and  $H_{\text{cpt}}^3(X) \cong H^3(X, X \setminus Z) \cong H^1(Z) = \{0\}$  by the duality theorem. Then we obtain the exact sequence,

$$0 \rightarrow H_{\text{cpt}}^2(X) \rightarrow H^2(X) \rightarrow H^2(S) \rightarrow 0.$$

We also have  $H_{\text{cpt}}^2(X) \cong H^2(X, X \setminus Z) \cong H^0(Z) \cong \mathbb{R}$ . Thus  $\dim H^2(X) = \dim H^2(S) + 1$ . If  $b_2(Z) = \dim H^2(Z) = \dim H^2(X)$  is greater than 1, it follows from the exact sequence that there is a Kähler class which does not belong to the compactly supported cohomology group. Thus in the cases, we obtain a new family of Ricci-flat Kähler metrics on  $K_Z$ .

**Example 6.4** Recently Futaki-Ono-Wang [10] obtained an Einstein-Sasakian metric on the sphere bundle of the canonical line bundle of every toric Fano manifold which gives the Ricci-flat Kähler cone metric on the complement  $K_Z \setminus \{0\}$ . Thus the theorem 5.1 shows that there exists a Ricci-flat conical Kähler metric in every Kähler class on the total space of the canonical line bundle on an arbitrary toric Fano manifold. If the canonical line bundle is the  $m$ -th tensor of a line bundle  $L$  on a toric Fano manifold, then we can apply the theorem to the total space  $L$  to obtain a Ricci-flat conical Kähler metric on  $L$  in each Kähler class. In the toric case, Futaki [9] constructed a Ricci-flat Kähler metric in the class of the compactly supported cohomology group. Let  $\widehat{\mathbb{CP}^2}$  be the blown up  $\mathbb{CP}^2$  at one point. Then  $\widehat{\mathbb{CP}^2}$  is a toric Fano manifold which does not admit a Kähler-einstein metric. However it is shown that the canonical line bundle of  $\widehat{\mathbb{CP}^2}$  admits a family of Ricci-flat conical Kähler metrics which is parametrized by the open set  $H^2(\widehat{\mathbb{CP}^2})$ , i.e., the Kähler cone of  $Z$ . Note that  $\dim H^2(\widehat{\mathbb{CP}^2}) = 2$  and  $\dim H_{\text{cpt}}^2(\widehat{\mathbb{CP}^2}) = 1$ . Oota-Yasui [22] described a Ricci-flat metric on a resolved Calabi-Yau cone whose asymptotic behavior seems to be different from the one in [9]. It is intriguing to show that these two Ricci-flat metrics are included in the two dimensional family constructed by the theorem 5.1. We denote by  $\Sigma_k$  a blown up  $\mathbb{CP}^2$  at  $k$  points in general position ( $0 \leq k \leq 8$ ). Then it is known that  $\Sigma_k$  admits an Einstein-Kähler metric for  $3 \leq k \leq 8$ . Since  $\Sigma_k$  for  $k = 1, 2$  is a toric Fano surface, there is a Einstein-Sasakian metric on the sphere bundle on  $\Sigma_k$ . Hence we obtain a complete Ricci-flat Kähler metrics in every Kähler class of the canonical line bundle of a blown up  $\mathbb{CP}^2$  at generic  $k$  points for all  $0 \leq k \leq 8$ .

**Example 6.5** Let  $X_0$  be the 3 dimensional hypersurface in  $\mathbb{C}^4$  defined by  $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$ . The hypersurface  $X_0$  has a singularity at the origin 0

which is the ordinary double point and there is a small resolution  $X \rightarrow X_0$  with trivial  $K_X$ . The small resolution  $X$  is the total space  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  on  $\mathbb{CP}^1$  and  $X_0$  is obtained by the contraction of the zero-section to one point. The complement  $X_0 \setminus \{0\}$  is the cone  $C(S)$  of the sphere bundle  $S$  over the product  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . Thus the sphere bundle  $S$  has an Einstein-Sasakian metric which induces the Ricci-flat Kähler cone metric on  $C(S)$ . Then we obtain a one dimensional family of Ricci-flat conical Kähler metrics on the resolution, since  $\dim H^2(X) = 1$  and  $H_{\text{cpt}}^2(X) = 0$ . Candelas and de la Ossa [6] described a Ricci-flat metric on a conifold and it is also keen to show that the Ricci-flat metric in [6] coincides with the one in our family constructed by the theorem 5.1.

## 7 Appendix

We shall show the following lemmas 7.2 and 7.3 to prove the proposition 4.2. Let  $\text{Ric}_{g_S}$  be the Ricci curvature of a Sasakian manifold  $(S, g_S)$ . We see that  $d\eta_S$  is a Kähler form on the distribution  $D$  which satisfies  $L_{\xi_S} d\eta_S = 0$ . Then  $d\eta_S$  gives the transversal Riemannian metric  $g_S^T$  on  $D$  with  $L_{\xi_S} g_S^T = 0$ . Let  $\text{Ric}_{g_S^T}$  be the Ricci curvature of the transversal Riemannian metric  $g_S^T$ . Then there is a relations between  $\text{Ric}_{g_S^T}$  and  $\text{Ric}_g$ ,

**Proposition 7.1** (*Boyer-Galicki, pp 224, theorem 7.3.12*)

$$\begin{aligned} \text{Ric}_g(u, v) &= \text{Ric}_{g_S^T}(u, v) - 2g(u, v), \quad u, v \in D \\ \text{Ric}_g(u, \xi) &= 2(n-1)\eta_S(u), \quad \forall u \in TS \end{aligned}$$

Thus a Sasakian metric  $g_S$  is Einstein if and only if the transversal metric  $g^T$  is Einstein with Einstein constant  $2n$ .

**Lemma 7.2** *Let  $(\Omega, \omega)$  be a weighted Calabi-Yau structure on  $C(S)$  and  $(\eta_S, \psi_S)$  the corresponding structure defined in (4.9) which induces a Sasakian structure on  $S$ . Let  $g_S^T$  be the transversal Riemannian metric given by the transversal Kähler structure  $d\eta_S$  on  $S$ . Then  $g_S^T$  is Einstein with Einstein constant  $2n$ , that is,  $\text{Ric}_{g_S^T} = 2ng_S^T$ .*

*proof* We take an open covering  $\{U_\alpha\}_\alpha$  of  $S$  such that each  $U_\alpha$  admits coordinates  $(x, z_1^\alpha, \dots, z_{n-1}^\alpha)$  which is compatible with transversal holomorphic structure on  $S$ , that is,  $(z_1^\alpha, \dots, z_{n-1}^\alpha)$  gives the transversal complex structures and  $\frac{\partial}{\partial x} = \xi_S$ . (Such coordinates are called foliation coordinates.) Then we have a  $\bar{\partial}$ -closed, local holomorphic  $n$ -form  $\Omega_{S,\alpha} = dz_1^\alpha \wedge \dots \wedge dz_{n-1}^\alpha$  which is a basic section of  $K_D|_{U_\alpha}$ . For simplicity, we write  $\Omega_{S,\alpha}$  for its pullback

$p_S^* \Omega_{S,\alpha}$  on  $C(S)$ . Since  $\psi = i_{\frac{\partial}{\partial t}} \Omega$  is a holomorphic section of  $p_S^* K_D$ , there is a holomorphic function  $f_\alpha$  on  $p_S^{-1}(U_\alpha)$  such that

$$\psi = e^{f_\alpha} \Omega_{S,\alpha}.$$

Substituting  $\psi = e^{f_\alpha} \Omega_{S,\alpha}$  into (4.13), we have

$$e^{f_\alpha + \bar{f}_\alpha} \eta \wedge \Omega_{S,\alpha} \wedge \sigma(\bar{\Omega}_{S,\alpha}) = c_{n-1} r^{2n} \eta \wedge \left(\frac{1}{2} d\eta\right)^{n-1} \quad (7.1)$$

We define a function  $k_\alpha$  locally by

$$\Omega_{S,\alpha} \wedge \sigma(\bar{\Omega}_{S,\alpha}) = e^{k_\alpha} c_{n-1} \left(\frac{1}{2} d\eta_S\right)^{n-1} \quad (7.2)$$

Then the Ricci form of the transversal Kähler form  $g_S^T$  on an open set  $U_\alpha$  of  $S$  is given by  $\text{Ric}_{g_S^T} = \sqrt{-1} \partial_B \bar{\partial}_B k_\alpha$ , where  $\bar{\partial}_B$  is the  $\bar{\partial}$ -operator with respect to the transversal complex structure on  $S$  and  $\partial_B$  is its complex conjugate. By multiplying  $\eta = \eta_S$  to both sides of (7.2) and pulling back by  $p_S$  to  $C(S)$ , we have

$$\eta \wedge \Omega_{S,\alpha} \wedge \sigma(\bar{\Omega}_{S,\alpha}) = e^{p_S^* k_\alpha} c_{n-1} \eta \wedge \left(\frac{1}{2} d\eta_S\right)^{n-1} \quad (7.3)$$

Comparing to (7.1), we have

$$p_S^* e^{k_\alpha} = r^{2n} e^{-(f_\alpha + \bar{f}_\alpha)} = e^{2nt} e^{-(f_\alpha + \bar{f}_\alpha)} \quad (7.4)$$

A function  $p_S^* \kappa_\alpha$  on  $C(S)$  is basic with respect to both  $\xi, \frac{\partial}{\partial t}$ . Thus we have  $p_S^* \partial_B \bar{\partial}_B k_\alpha = \partial \bar{\partial} p_S^* \kappa_\alpha$ . The it follows that

$$\text{Ric}_{g_S^T} = \sqrt{-1} \partial_B \bar{\partial}_B k_\alpha = i_S^* \sqrt{-1} \partial \bar{\partial} p_S^* \kappa_\alpha,$$

where  $\bar{\partial}$  is the  $\bar{\partial}$ -operator with respect to the complex structure on  $C(S)$ . From (7.4), we have

$$\text{Ric}_{g_S^T} = 2ni_S^* \sqrt{-1} \partial \bar{\partial} t - i_S^* \sqrt{-1} \partial \bar{\partial} (f_\alpha + \bar{f}_\alpha) \quad (7.5)$$

Since  $\bar{\partial} f_\alpha = 0$ , we have  $\partial \bar{\partial} (f_\alpha + \bar{f}_\alpha) = 0$  and the transversal Kähler form is  $\frac{1}{2} d\eta_S = \sqrt{-1} \partial \bar{\partial} t|_S$ . Thus we obtain

$$\text{Ric}_{g_S^T} = 2n \sqrt{-1} \partial \bar{\partial} t = 2n \left(\frac{1}{2} d\eta_S\right) \quad (7.6)$$

Then it follows that  $\text{Ric}_{g_S^T} = 2n g_S^T$ . q.e.d.

Conversely the following lemma shows that an Einstein-Sasakian structure on  $S$  gives the weighted Calabi-Yau structure on  $C(S)$ ,

**Lemma 7.3** *An Einstein-Sasakian structure on  $S$  corresponds to the weighted Calabi-Yau structure on  $C(S)$  under the correspondence in the proposition 3.6*

*proof* As in the proposition 7.1, an Einstein-Sasaki structure gives the transversal Einstein-Kähler metric with scalar curvature  $2n$ ,  $\text{Ric}_{g_S^T} = 2ng_S^T$ . Thus the 1-st Chern class  $c_B^1(S)$  of the transversal canonical line bundle  $K_D$  is represented by the form  $2n(\frac{1}{2}d\eta_S) = n d\eta_S$ . There is an exact sequence on  $S$ ,

$$\rightarrow H^0(S) \xrightarrow{i} H_B^2(S) \xrightarrow{j} H^2(S) \rightarrow$$

in which we have  $i(\alpha) = \alpha d\eta_S$  ( $\alpha \in H^0(S)$ ) and  $j(c_B^1(S)) = nj(d\eta_S) = i \circ j(n) = 0$ . Thus  $c^1(S) = c^1(K_D)$  vanishes. Since  $S$  is simply connected, the line bundle  $K_D$  is trivial. Let  $\{U_\alpha\}$  be an open covering of  $S$  such that each  $U_\alpha$  admits a foliation coordinates  $(x^\alpha, z_1^\alpha, \dots, z_{n-1}^\alpha)$ . We denote by  $p_s : C(S) \rightarrow S$  the projection. We take  $U_\alpha$  sufficiently small such that every  $d$ -closed holomorphic 1-form on  $p_s^{-1}(U_\alpha) \cong \mathbb{R} \times U_\alpha$  is written as a  $d$ -exact 1-form of a holomorphic function, that is, the holomorphic de Rham theorem holds on  $U_\alpha$ . Taking a  $d$ -closed, basic, holomorphic  $(n-1)$ -form  $\Omega_{S,\alpha} = dz_1^\alpha \wedge \dots \wedge dz_{n-1}^\alpha$  on  $U_\alpha$ , we define a basic function  $k_\alpha$  by

$$\Omega_{S,\alpha} \wedge \sigma(\overline{\Omega}_{S,\alpha}) = c_{n-1} e^{\kappa_\alpha} (d\eta_S)^{n-1} \quad (7.7)$$

Then the transversal Ricci form  $\text{Ric}_{g_S^T}$  is given by

$$\text{Ric}_{g_S^T} = \sqrt{-1} \partial_B \overline{\partial}_B \kappa_\alpha \quad (7.8)$$

Since the transversal Kähler form  $\frac{1}{2}d\eta_S$  is Einstein-Kähler, we have

$$\sqrt{-1} \partial_B \overline{\partial}_B \kappa_\alpha = 2n(\frac{1}{2}d\eta_S) = 2ni_s^*(\sqrt{-1} \partial \overline{\partial} t) \quad (7.9)$$

Pulling back them by the projection  $p_s : C(S) \rightarrow S$  to  $C(S)$ , since  $\kappa_\alpha$  is a basic function, we have

$$p_s^* \text{Ric}_T(\frac{1}{2}d\eta_S) = p_s^* \sqrt{-1} \partial_B \overline{\partial}_B \kappa_\alpha \quad (7.10)$$

$$= \sqrt{-1} \partial \overline{\partial} p_s^* \kappa_\alpha = 2n(\sqrt{-1} \partial \overline{\partial} t) \quad (7.11)$$

Then on  $p_s^{-1}(U_\alpha) \subset C(S)$ , we have

$$\sqrt{-1} \partial \overline{\partial} (p_s^* \kappa_\alpha - 2nt) = 0$$

Since  $U_\alpha$  is sufficiently small such that every  $d$ -closed holomorphic 1-form is a  $d$  exact form of a holomorphic function, there is a holomorphic function  $q_\alpha$  on  $p^{-1}(U_\alpha)$  satisfying

$$\kappa_\alpha - 2nt = q_\alpha + \bar{q}_\alpha$$

Then from (7.7), we have

$$(e^{-q_\alpha} \Omega_{S,\alpha}) \wedge \sigma(e^{-\bar{q}_\alpha} \bar{\Omega}_{S,\alpha}) = c_{n-1} r^{2n} \left(\frac{1}{2} d\eta_S\right)^{n-1} \quad (7.12)$$

Note that  $r = e^t$ . We define  $\psi$  by  $e^{-q_\alpha} \Omega_{S,\alpha}$ . Then the pair  $(\eta_S, \psi)$  satisfies (4.14). If we set  $\Omega_\alpha = (dt + \sqrt{-1}Jdt) \wedge (e^{-q_\alpha} \Omega_{S,\alpha})$ , then  $\Omega_\alpha$  is a local holomorphic  $n$ -form on  $C(S)$  which satisfies

$$\Omega_\alpha \wedge \sigma(\bar{\Omega}_\alpha) = c_n \omega^n$$

where  $c_n = \frac{(2\sqrt{-1})^n}{n!}$ . Hence the Ricci curvature of the Kähler metric  $\omega$  on  $C(S)$  vanishes. Further since  $S$  is simply connected, the canonical line bundle on  $C(S)$  is trivial. Then there is a holomorphic  $n$ -form  $\Omega$  on  $C(S)$  satisfying the Monge-Ampère equation,

$$\Omega \wedge \sigma(\bar{\Omega}) = c_n \omega^n$$

As we see in the section 1,

$$\mathbb{L}_{\frac{\partial}{\partial t}} J = 0, \quad \mathbb{L}_{\frac{\partial}{\partial t}} \omega = 2\omega$$

Since  $\frac{\partial}{\partial t}$  is the real part of a holomorphic vector field on  $C(S)$ , there is a holomorphic function  $h$  such that

$$\mathbb{L}_{\frac{\partial}{\partial t}} \Omega = h\Omega$$

By the action of the Lie derivative  $\mathbb{L}_{\frac{\partial}{\partial t}}$  on both sides of the Monge-Ampère equation, we have

$$(k + \bar{k})\Omega \wedge \sigma(\bar{\Omega}) = 2nc_n \omega^n$$

It follows that  $k + \bar{k} = 2n$ . Since  $k$  is holomorphic, it follows that  $k$  is a constant  $n$ . Hence

$$\mathbb{L}_{\frac{\partial}{\partial t}} \Omega = n\Omega$$

Thus  $(\Omega, \omega)$  is a weighted Calabi-Yau structure on  $C(S)$ . Hence we have the result. q.e.d.

*proof of theorem 4.2* The theorem follows from the lemmas 7.2 and 7.3 and the proposition 7.1. q.e.d.

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